

ALMOST α -PARACOSYMPLECTIC MANIFOLDS

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ABSTRACT. This paper is a complete study of almost α -paracosmplectic manifolds. We characterize almost α -paracosmplectic manifolds which have para Kaehler leaves. Main curvature identities which are fulfilled by any almost α -paracosmplectic manifold are found. We also proved that ξ is a harmonic vector field if and only if it is an eigen vector field of the Ricci operator. We locally classify three dimensional almost α -para-Kenmotsu manifolds satisfying a certain nullity condition. We show that this condition is invariant under $D_{\gamma,\beta}$ -homothetic deformation. Furthermore, we construct examples of almost α -paracosmplectic manifolds satisfying generalized nullity conditions.

1. Introduction

The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [23] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy [40]. However such structures were studied before [36], [5], [6]. Note also [2]. These authors called such structures almost para-coHermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [14], [39], [40].

Considering the recent stage of the theory development there is an impression that the geometers are focused on problems in almost paracontact metric geometry which are seem to be created ad hoc, but in fact the source for them lies in the Riemannian geometry of almost contact metric structures. The basic reference for almost contact metric manifolds is a D. E. Blair monograph [3]. Recently appeared long awaited a survey article [9] concernig almost cosymplectic manifolds as the Blair's monograph deals mostly with contact metric manifolds.

Both almost contact metric and almost paracontact metric manifolds have common roots in something we may call pre-cosymplectic structure which simply is a pair of a 1-form usually denoted by η and 2-form Φ , so $\eta \wedge \Phi^n$ is a volume element. The characteristic (Reeb) vector field ξ is then defined by $i_\xi \eta = 1$, $i_\xi \Phi = 0$. The Riemannian or pseudo-Riemannian geometry in this framework appears when one is trying to introduce a *compatible* structure which means a metric or pseudo-metric g and an affinor ϕ ((1,1)-tensor field), such that $\Phi(X, Y) = g(X, \phi Y)$, and $\phi^2 = \epsilon(Id - \eta \otimes \xi)$, where for $\epsilon = -1$ we have almost contact metric structure and for $\epsilon = +1$ almost paracontact metric structure. The triple (ϕ, ξ, η) is then called almost contact structure or almost paracontact structure, resp. For example: when η is a contact form $d\eta = \Phi$ manifolds are called contact metric or paracontact metric, for both η, Φ closed, P. Libermann called such pair a cosymplectic structure, we have almost cosymplectic manifolds or almost paracosymplectic manifolds.

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The other possible point of view is to take an almost contact or almost paracontact structures as a starting point and next to seek a compatible metric or pseudo-metric.

Combining the assumption concernig the forms η , Φ and the affnir ϕ we obtain several disjoint (rough) classes of manifolds. Additionally within each of these classes are posed some assumptions concernig the metric or pseudo-metric. Even if almost paracontact metric manifolds were studied in the past it is recently when geometers discovered many similarities between Riemannian and pseudo-Riemannian geometry of almost contact metric and almost paracontact metric manifolds. Up to the level when we can simply transliterate some properties.

Also this paper deals with the concept well-known in almost contact metric geometry: manifolds with Reeb field belonging the the κ -nullity ditribution and more general (κ, μ) -nullity or even (κ, μ, ν) -nullity distributions, here κ , μ , ν are constants or particular functions. Classifications are obtained for non-Sasakian contact metric manifold, almost cosymplectic, almost α -Kenmotsu and almost α -cosymplectic, [4], [11], [16], [20], [32], [33], [37].

The similar problems are now posed and studied for an almost paracontact metric manifolds. However the situation is more difficult according to the fact that occurs “exceptional” manifolds, that means manifolds without counterparts in the Riemannian case e.g. [15], [28].

These “exceptions” are often contradites our intuitions. Also when thinking about tight relation between topology of a manifold and its Riemannian geometry, particularly for closed manifolds, from other hand pseudo-Riemannian metric are rather loosely related to the manifold’s topology we see that some problems can not be simply brought from the almost contact metric geometry to almost paracontact.

Summarizing the contents of this paper, after the Preliminaries, where we recall the definition of almost paracontact metric manifold, we introduce a class of manifolds which contains both almost paracosymplectic and almost para-Kenmotsu as well and we call these manifolds as almost α -paracosymplectic, where α is a arbitrary function. However we prove later on that in fact if dimension of the manifold is ≥ 5 , then the 1-forms $d\alpha$ and η are proportional.

There are basic objects for arbitrary almost paracontact metric manifold: tensor fields $\mathcal{A} = -\nabla\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi\phi$. We study basic relations between them for the case of almost α -paracosymplectic manifold. It is also established that ξ is geodesic and ϕ is ξ -parallel, $\nabla_\xi\phi = 0$.

In the short auxiliary section we recall the concept of para-Kaehler manifolds we need to define a class of almost α -paracosymplectic manifolds with para-Kaehler leaves.

In the Sect. 5. we characterize manifolds with para-Kaehler leaves: an almost α -paracosymplectic manifold has para-Kaehler leaves if and only if

$$(\nabla_X\phi)Y = \alpha g(\phi X, Y)\xi + g(hX, Y)\xi - \alpha\eta(Y)\phi X - \eta(Y)hX.$$

In the Sect. 6. we determine $U(X, Y) = (\nabla_{\phi X}\phi) - (\nabla_X\phi)Y$ and other equivalent forms. One of the most important object is a vector valued 2-form Ω , defined as $\Omega(X, Y) = R(X, Y)\xi$. We give its form in this section. Note that Ω is more complicated for the case $(M) = 3$ and $\alpha \neq \text{const}$. There is a difference between 3- and higher-dimensional manifolds for α non-constant.

The Sect. 7. is particularly devoted to almost α -paracosymplectic manifolds with α const. Such manifolds are also known as almost α -para-Kenmotsu manifolds and we follow this terminolgy to emphasize that $\alpha = \text{const}$. In this section we obtain some curvature identities for such manifolds. Also we provide more detailed study of the Jacobi

operator $LX = R(X, \xi)\xi$ and related objects. When manifold has para-Kaehler leaves we measure the commutator $Q\phi - \phi Q$ with the Ricci operator Q . Finally we notice that manifold with h vanishing everywhere has a simple local structure of a warped product $\mathbb{R} \times_f M$ of real line and almost para-Kaehler manifold.

When equip the tangent bundle of the manifold with a metric we can study the problem of the “harmonicity” of the characteristic vector field ξ , where we consider ξ as a map between the manifold and its tangent bundle. For an almost α -paracosymplectic manifolds ξ is harmonic if and only if it is an eigenvector field of the Ricci operator, $Q\xi = f\xi$. This is proved in the Sect. 8.

In the Sect. 9. it is proved that an almost α -paracosymplectic manifold of dimension ≥ 5 is locally conformal to almost para-cosymplectic manifold and is locally $D_{1,\alpha}$ -homothetic to almost para-Kenmotsu manifold near the points where $\alpha \neq 0$.

In the Sect. 10. there are considered so-called almost α -para Kenmotsu (κ, μ, ν) -spaces. These manifolds are depicted by the requirement that the form $R(X, Y)\xi$ is uniquely determined by the respective Jacobi operator $LX = R(X, \xi)\xi$ in the way that $R(X, Y)\xi = \eta(Y)LX - \eta(X)LY$. Then we assume that l has very particular shape $l = \kappa\phi^2 + \mu h + \nu\phi h$, κ, μ, ν are constants or more generally functions however rather particular. The main result in this section is that all these manifolds have para-Kahler leaves.

Finally in the last section we classify locally 3-dimensional almost α -para Kenmotsu manifolds studying possible canonical forms for the tensor field h . As an application we describe the corresponding Ricci operators. In this way it is discovered the connection between 3-manifolds with harmonic characteristic vector field and (κ, μ, ν) -spaces: if ξ is harmonic vector field then M locally has a structure of (κ, μ, ν) -space, conversely for 3-dimensional (κ, μ, ν) -space the characteristic vector field is harmonic.

2. PRELIMINARIES

Let M be a $(2n+1)$ -dimensional differentiable manifold and ϕ is a $(1, 1)$ tensor field, ξ is a vector field and η is a one-form on M . Then (ϕ, ξ, η) is called an almost paracontact structure on M if

$$(i) \quad \eta(\xi) = 1, \quad \phi^2 = Id - \eta \otimes \xi,$$

(ii) the tensor field ϕ induces an almost paracomplex structure on the distribution $D = \ker \eta$, that is the eigendistributions D^\pm , corresponding to the eigenvalues ± 1 , respectively have equal dimensions, $\dim D^+ = \dim D^- = n$. The manifold M is said to be almost paracontact manifold if it is endowed with an almost paracontact structure [40].

Let M be an almost paracontact manifold. M will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$(2.1) \quad g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y).$$

For such manifold, we additionally have

$$(2.2) \quad \eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0.$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$(2.3) \quad \Phi(X, Y) = g(\phi X, Y),$$

usually called a fundamental form corresponding to the structure. For an almost α -paracosymplectic manifold, there always exists an orthogonal basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$

such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \dots, n\}$. Such basis is called a ϕ -basis.

On an almost paracontact manifold, one defines the $(2, 1)$ -tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be normal [40]. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$

$$J(X, \lambda \frac{d}{dt}) = (\phi X + \lambda \xi, \eta(X) \frac{d}{dt}),$$

is integrable.

3. ALMOST α -PARACOSYMPLECTIC MANIFOLDS

An almost paracontact metric manifold M^{2n+1} , with a structure (ϕ, ξ, η, g) is said to be an almost α -paracosymplectic manifold if the form η is closed and $d\Phi = 2\alpha\eta \wedge \Phi$, where α may be a constant or a function on M . Although α is arbitrary we will prove that if dimension $d = 2n + 1$ of M is ≥ 5 , then $d\alpha = f\eta$ for a (smooth) function f .

For a particular choices of the function α we have the following classes of manifolds

- almost α -para-Kenmotsu manifolds

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi, \quad \alpha = \text{const.},$$

- normal almost α -para-Kenmotsu manifolds are called α -para-Kenmotsu,
- almost paracosymplectic

$$d\eta = 0, \quad d\Phi = 0,$$

quite similar normal almost paracosymplectic manifolds are paracosymplectic.

It is clear that almost 0-para-Kenmotsu manifold is an almost paracosymplectic manifold.

In what will follow we establish the fundamental properties of the structure's tensor fields.

Definition 1. For an almost α -paracosymplectic manifold, define the $(1, 1)$ -tensor field \mathcal{A} by

$$(3.1) \quad \mathcal{A}X = -\nabla_X \xi.$$

Proposition 1. For an almost α -paracosymplectic manifold M^{2n+1} , we have

$$(3.2) \quad \begin{aligned} i) \mathcal{L}_\xi \eta &= 0, \quad ii) g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y), \quad iii) \mathcal{A}\xi = 0, \\ iv) \mathcal{L}_\xi \Phi &= 2\alpha\Phi, \quad v) (\mathcal{L}_\xi g)(X, Y) = -2g(\mathcal{A}X, Y), \\ vi) \eta(\mathcal{A}X) &= 0, \quad vii) d\alpha = f\eta \text{ if } n \geq 2 \end{aligned}$$

where \mathcal{L} indicates the operator of the Lie differentiation and X is an arbitrary vector field on M^{2n+1} .

Proof. To prove *i)* and *iv)* we use the coboundary formula

$$\mathcal{L}_\xi \eta = d \circ i_\xi \eta + i_\xi \circ d\eta,$$

for the Lie derivative acting on skew-forms. We note that $i_\xi \eta = 1$ and $d\eta = 0$. Similarly $(i_\xi \Phi)(X) = \Phi(\xi, X) = 0$ for an arbitrary vector field, hence $i_\xi \Phi = 0$. Finally

$$(3.3) \quad \mathcal{L}_\xi \Phi = i_\xi d\Phi = i_\xi(2\alpha\eta \wedge \Phi) = 2\alpha(i_\xi \eta \wedge \Phi - \eta \wedge i_\xi \Phi) = 2\alpha\Phi$$

Note $2d\eta(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = -g(\mathcal{A}X, Y) + g(X, \mathcal{A}Y)$, where the last equality follows from the definition of \mathcal{A} . As η is closed \mathcal{A} is symmetric (or self-adjoint), we completed the proof of *ii*). Using the definition of Lie differentiation and \mathcal{A} , we obtain

$$(3.4) \quad (\mathcal{L}_\xi g)(X, Y) = \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y])$$

$$(3.5) \quad = -g(\mathcal{A}X, Y) - g(X, \mathcal{A}Y) = -2g(\mathcal{A}X, Y).$$

(3.5) implies *v*). For ξ is unit vector field we have for arbitrary vector field X , $0 = Xg(\xi, \xi) = 2g(\nabla_X \xi, \xi) = -2\eta(AX) = -2g(A\xi, X)$ which yield *iii*) and *vi*). Finally to proof *vii*) we need the following \square

Lemma 1. *Let ω be a 2-form on a manifold \bar{M} , $\dim(\bar{M}) = n \geq 4$ and ω has maximal rank at every point, equivalently $\omega^{\wedge[\frac{n}{2}]}$ is non-zero at every point. If for a 1-form β on \bar{M} , $\beta \wedge \omega = 0$ at a point $p \in \bar{M}$, then $\beta = 0$ at p . Particularly β vanishes everywhere on \bar{M} if $\beta \wedge \omega$ is everywhere zero.*

Proof. Let $\beta \wedge \omega = 0$ at p and $\beta_p \neq 0$. Then there is a vector v at p , such that $\beta_p(v) = 1$ and $i_v(\beta \wedge \omega)_p = \omega_p - \beta_p \wedge \gamma_p$, $\gamma_p = i_v \omega_p$. Hence $\omega_p = \beta_p \wedge \gamma_p$ and $\omega_p^{\wedge 2} = 0$. In consequence as $[\frac{n}{2}] \geq 2$, $\omega_p^{\wedge[\frac{n}{2}]} = 0$ which contradicts our assumption that ω is of maximal rank.

Now we are going back to the proof of the part *vii*). We put $\beta = 2\alpha\eta$. So $d\Phi = \beta \wedge \Phi$, applying exterior differential to this equation and taking interior product with i_ξ in the result, we obtain $0 = \gamma \wedge \Phi$ ($i_\xi \Phi = 0$) everywhere, $\gamma = i_\xi d\beta$. If $\dim(M^{2n+1}) \geq 5$ ($n \geq 2$) by the above Lemma γ vanishes identically on M^{2n+1} . Notice $\gamma = i_\xi d\beta = 2i_\xi(d\alpha \wedge \eta)$ as $d\eta = 0$ and $0 = (i_\xi d\alpha)\eta - d\alpha$, ($i_\xi \eta = 1$). Hence $d\alpha = f\eta$, $f = i_\xi d\alpha$. \square

Proposition 2. *For an almost α -paracosymplectic manifold, we have*

$$(3.6) \quad \mathcal{A}\phi + \phi\mathcal{A} = -2\alpha\phi, \quad \nabla_\xi \phi = 0.$$

Proof. $(\mathcal{L}_\xi \Phi)(X, Y) = \xi\Phi(X, Y) - \Phi([\xi, X], Y) - \Phi(X, [\xi, Y])$ the definition of Φ follows

$$\begin{aligned} (\mathcal{L}_\xi \Phi)(X, Y) &= \xi g(\phi X, Y) - g(\phi[\xi, X], Y) - g(\phi X, [\xi, Y]) \\ &= g((\nabla_\xi \phi)X - \phi\mathcal{A}X - \mathcal{A}\phi X, Y). \end{aligned}$$

We already know $\mathcal{L}_\xi \Phi = 2\alpha\Phi$, therefore these both identities yield

$$2\alpha\phi X = (\nabla_\xi \phi)X - \phi\mathcal{A}X - \mathcal{A}\phi X.$$

We have $\nabla_\xi \phi^2 = \nabla_\xi(Id - \eta \otimes \xi) = 0$ for both $\nabla_\xi \eta$ and $\nabla_\xi \xi$ vanish identically. From other hand we have

$$(\nabla_\xi \phi^2)X = \phi(\nabla_\xi \phi)X + (\nabla_\xi \phi)\phi X.$$

Hence $\phi(\nabla_\xi \phi)X = -(\nabla_\xi \phi)\phi X$ and if $\phi X = X$, that is X is a field of eigenvectors corresponding to $+1$ -eigenvalue ($[+1]$ -vector field), then

$$2\alpha X = (\nabla_\xi \phi)X - \phi\mathcal{A}X - \mathcal{A}X,$$

applying ϕ to the both hands we get

$$2\alpha X = \phi(\nabla_\xi \phi)X - \phi^2 \mathcal{A}X - \phi\mathcal{A}X = -(\nabla_\xi \phi)X - \mathcal{A}X - \phi\mathcal{A}X,$$

and these both above identities follow $(\nabla_\xi \phi)X = -(\nabla_\xi \phi)X = 0$. The same arguments prove $(\nabla_\xi \phi)X = 0$ for $[-1]$ -vector field $\phi X = -X$. Obviously $(\nabla_\xi \phi)\xi = \nabla_\xi \phi\xi - \phi\nabla_\xi \xi = 0$.

Therefore $\nabla_\xi \phi = 0$ identically as near each point there is a frame of vector fields consisting only from ξ and eigenvector fields of ϕ . \square

Let define $h = \frac{1}{2}\mathcal{L}_\xi \phi$. In the following proposition we establish some properties of the tensor field h .

Proposition 3. *For an almost α -paracosymplectic manifold, we have the following relations*

$$(3.7) \quad g(hX, Y) = g(X, hY),$$

$$(3.8) \quad h \circ \phi + \phi \circ h = 0,$$

$$(3.9) \quad h\xi = 0,$$

$$(3.10) \quad \nabla \xi = \alpha\phi^2 + \phi \circ h = -\mathcal{A}.$$

Proof. Similarly as in the Proposition 2 we have

$$(3.11) \quad (\mathcal{L}_\xi \phi^2)X = \phi(\mathcal{L}_\xi \phi)X + (\mathcal{L}_\xi \phi)\phi X = 2\phi hX + 2h\phi X.$$

and

$$(3.12) \quad \mathcal{L}_\xi \phi^2 = -(\mathcal{L}_\xi \eta) \otimes \xi = 0.$$

From (3.11) and (3.12) we get (3.8). By using the formula $(\mathcal{L}_\xi \phi)X = [\xi, \phi X] - \phi[\xi, X] = \nabla_\xi \phi X - \nabla_{\phi X} \xi - \phi(\nabla_\xi X - \nabla_X \xi)$ we obtain

$$(3.13) \quad h = \frac{1}{2}(\mathcal{A}\phi - \phi\mathcal{A}).$$

The last formula and the properties of ϕ and \mathcal{A} (symmetry) follow that h is also a symmetric tensor field, $g(hX, Y) = g(X, hY)$. Moreover $h\xi = 0$ and $\eta \circ h = 0$. Using (3.3), (3.5) and the following identity

$$(\mathcal{L}_\xi \Phi)(X, Y) = (\mathcal{L}_\xi g)(\phi X, Y) + g((\mathcal{L}_\xi \phi)X, Y),$$

we obtain

$$(3.14) \quad \alpha\phi = -\mathcal{A}\phi + h.$$

If we apply ϕ from the right to the (3.14) and use the anticommutative h and ϕ , we have

$$\alpha\phi^2 + \phi \circ h = -\mathcal{A} = \nabla \xi.$$

\square

Corollary 1. *All the above Propositions imply the following formulas for the traces*

$$(3.15) \quad \begin{aligned} \text{tr}(\mathcal{A}\phi) &= \text{tr}(\phi\mathcal{A}) = 0, \quad \text{tr}(h\phi) = \text{tr}(\phi h) = 0, \\ \text{tr}(\mathcal{A}) &= -2\alpha n, \quad \text{tr}(h) = 0. \end{aligned}$$

4. PARA-KAEHLER MANIFOLDS

This is an auxiliary section. The general reference for the notions which appear here is [13]. We recall here basic concepts of a para-Hermitian geometry. An even dimensional manifold M^{2n} endowed with a pair an almost para-Hermitian structure (J, \langle, \rangle) , where J is an almost para-complex structure and \langle, \rangle is a pseudo-Riemannian metric. These tensor fields are subject of the following conditions

$$J^2 = Id, \quad \langle JX, JY \rangle = -\langle X, Y \rangle,$$

as it is common X, Y denote vector fields. The manifold M endowed with this structure is called an almost para-Hermitian manifold. The almost para-Hermitian manifold M is

para-Kaehler if the almost para-complex structure J is a covariant constant $\nabla J = 0$, with respect to the Levi-Civita connection. An almost para-complex structure is integrable if and only if the Nijenhuis torsion of J vanishes identically

$$N_J(X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY] = 0.$$

An almost para-complex structure of a para-Kaehler manifold is always integrable. In the terms of the local coordinates maps, integrability is equivalent to the existence of a set of maps, covering the manifold, the para-complex structure has constant coefficients in the local map coordinates. If $p \in M$ is a point, then near p we have coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$, the local components $J_i^k = \text{const.}$ are constants.

5. ALMOST α -PARACOSYMPLECTIC MANIFOLDS WITH PARA-KAEHLER LEAVES

The idea is to restrict further our consideration to the particular class of manifolds. However this class of manifolds is wide enough to provide interesting results and examples. In fact each 3-dimensional manifold belongs to this class. Let $M^{2n+1} = (M, \phi, \xi, \eta, g)$ be an almost α -paracosymplectic manifold. By the definition the form η is closed therefore a distribution $\mathcal{D} : \eta = 0$ is completely integrable. \mathcal{D} defines a foliation \mathcal{F} . Each leaf carries an almost para-Kaehler structure (J, \langle, \rangle)

$$J\bar{X} = \phi\bar{X}, \quad \langle \bar{X}, \bar{Y} \rangle = g(\bar{X}, \bar{Y}),$$

\bar{X}, \bar{Y} are vector fields tangent to the leaf. If this structure is para-Kaehler, leaf is called a para-Kaehler leaf of the manifold M .

Lemma 2. *An almost α -paracosymplectic manifold M has para-Kaehler leaves if and only if*

$$(\nabla_X \phi)Y = g(\mathcal{A}X, \phi Y)\xi + \eta(Y)\phi\mathcal{A}X, \quad \mathcal{A} = -\nabla\xi.$$

Proof. Let \mathcal{F}_a be a leaf passing through a point $a \in M$. The characteristic vector field is a normal vector field to \mathcal{F}_a , the restriction $\mathcal{A}|_{\mathcal{F}} = -\nabla\xi|_{\mathcal{F}}$ is the Weingarten operator (the shape tensor). The Gauss equation

$$\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + II(\bar{X}, \bar{Y})\xi,$$

yields

$$\begin{aligned} (\nabla_{\bar{X}}\phi)\bar{Y} &= \nabla_{\bar{X}}\phi\bar{Y} - \phi\nabla_{\bar{X}}\bar{Y} \\ (5.1) \quad &= \bar{\nabla}_{\bar{X}}J\bar{Y} + II(\bar{X}, \phi\bar{Y})\xi \\ &\quad - \phi(\bar{\nabla}_{\bar{X}}\bar{Y} + II(\bar{X}, \bar{Y})\xi) \\ &= (\bar{\nabla}_{\bar{X}}J)\bar{Y} + II(\bar{X}, \phi\bar{Y})\xi = II(\bar{X}, \phi\bar{Y})\xi \end{aligned}$$

here by assumption $\bar{\nabla}J = 0$ identically, II is the second fundamental form of \mathcal{F} , $II(\bar{X}, \bar{Y}) = g(-\nabla_{\bar{X}}\xi, \bar{Y})$. The above identity implies $(\nabla_X \phi)Y = g(\mathcal{A}X, \phi Y)\xi$ for arbitrary vector fields on the manifold M such that $\eta(X) = \eta(Y) = 0$. For arbitrary X, Y we have a decomposition $X = (X - \eta(X)\xi) + \eta(X)\xi$. To finish the proof we need to remind that $\nabla_\xi \phi = 0$ and $(\nabla_X \phi)\xi = \phi\mathcal{A}X$. \square

Proposition 4. *Let $M^{2n+1} = (M, \phi, \xi, \eta, g)$ be an almost α -paracosymplectic manifold. Then the foliation \mathcal{F} , when $\alpha = 0$ (resp. $\alpha \neq 0$), \mathcal{F} is totally geodesic (resp. totally umbilical) if and only if $h = 0$.*

Proof. Using Gauss equation we have $II(\bar{X}, \bar{Y}) = g(\nabla_{\bar{X}}\bar{Y}, \xi) = -g(\bar{Y}, \nabla_{\bar{X}}\xi) = -g(\bar{Y}, \alpha\phi^2\bar{X} + \phi h\bar{X}) = -\alpha g(\bar{X}, \bar{Y}) - g(\bar{X}, \phi h\bar{Y})$ for all $\bar{X}, \bar{Y} \in \Gamma(D)$. This completes proof. \square

Proposition 5. *An almost α -paracosymplectic manifold M has para-Kaehler leaves if and only if*

$$(5.2) \quad (\nabla_X \phi)Y = \alpha g(\phi X, Y)\xi + g(hX, Y)\xi - \alpha \eta(Y)\phi X - \eta(Y)hX$$

for $\alpha = 0$ it is a formula known for almost paracosymplectic manifolds.

Proof. If we use the Lemma 2 and the identity (3.10), we have

$$\begin{aligned} (\nabla_X \phi)Y &= -g(\alpha \phi^2 X + \phi hX, \phi Y)\xi - \eta(Y)\phi(\alpha \phi^2 X + \phi hX) \\ &= -\alpha g(\phi^2 X, \phi Y)\xi - g(\phi hX, \phi Y)\xi - \alpha \eta(Y)\phi X - \eta(Y)hX. \end{aligned}$$

By the help of (2.1) we get the requested equation. \square

As a direct consequence we have the following

Theorem 1. *Let M^{2n+1} be an almost α -para-Kenmotsu manifold with para Kaehler leaves. Then M^{2n+1} is a para-Kenmotsu ($\alpha = 1$) manifold if and only if $\mathcal{A} = -\phi^2$.*

Remark 1. *For a similar notion in contact metric geometry see e.g. [30], [20], [32] and there are many other papers where this notion appears explicitly or implicitly. Compare the references in [9]. We also note that in almost contact metric geometry there is more general idea of when a manifold additionally carries a so-called CR-structure. All almost contact metric manifolds with Kaehler leaves are also Levi-flat CR-manifolds.*

6. BASIC STRUCTURE AND CURVATURE IDENTITIES

Lemma 3. *For an almost α -paracosymplectic manifold (M, ϕ, ξ, η, g) with its fundamental 2-form Φ the following equations hold*

$$(6.1) \quad (\nabla_X \Phi)(Y, Z) = g((\nabla_X \phi)Y, Z),$$

$$(6.2) \quad (\nabla_X \Phi)(Z, \phi Y) + (\nabla_X \Phi)(Y, \phi Z) = -\eta(Y)g(\mathcal{A}X, Z) - \eta(Z)g(\mathcal{A}X, Y),$$

$$(6.3) \quad (\nabla_X \Phi)(\phi Y, \phi Z) - (\nabla_X \Phi)(Y, Z) = \eta(Y)g(\mathcal{A}X, \phi Z) - \eta(Z)g(\mathcal{A}X, \phi Y),$$

where $\mathcal{A} = -\nabla \xi$.

Proof. The proof of (6.1) is obvious. Differentiating the identity $\phi^2 = I - \eta \otimes \xi$ covariantly, we obtain

$$(6.4) \quad (\nabla_X \phi)\phi Y + \phi(\nabla_X \phi)Y = g(Y, \mathcal{A}X)\xi + \eta(Y)\mathcal{A}X.$$

If we take the inner product with Z , we obtain (6.2). Replacing Z by ϕZ in (6.2), using the anti-symmetry of Φ and (6.1), we get (6.3). \square

Proposition 6. *For any almost α -paracosymplectic manifold, we have*

$$(6.5) \quad (\nabla_{\phi X} \phi)\phi Y - (\nabla_X \phi)Y - \eta(Y)\mathcal{A}\phi X - 2\alpha(g(X, \phi Y)\xi + \eta(Y)\phi X) = 0.$$

Proof. Let us define $(0, 3)$ -tensor field \mathcal{B} as follows

$$\mathcal{B}(X, Y, Z) = g((\nabla_{\phi X} \phi)\phi Y, Z) - g((\nabla_X \phi)Y, Z) - \eta(Y)g(\mathcal{A}\phi X, Z) - 2\alpha(g(X, \phi Y)\eta(Z) + \eta(Y)g(\phi X, Z)).$$

Antisymmetrizing \mathcal{B} with respect to X, Y we have

$$\begin{aligned} \mathcal{B}(X, Y, Z) - \mathcal{B}(Y, X, Z) &= (\nabla_{\phi X} \Phi)(\phi Y, Z) - (\nabla_{\phi Y} \Phi)(\phi X, Z) \\ &\quad - (\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(X, Z) \\ &\quad - \eta(Y)g(\mathcal{A}\phi X, Z) + \eta(X)g(\mathcal{A}\phi Y, Z) \\ &\quad - 2\alpha((g(X, \phi Y) - g(Y, \phi X))\eta(Z) \\ &\quad + \eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)). \end{aligned} \tag{6.6}$$

Recalling the well known formula

$$\begin{aligned} 3d\Phi(X, Y, Z) &= (\nabla_X \Phi)(Y, Z) + (\nabla_Z \Phi)(X, Y) + (\nabla_Y \Phi)(Z, X) \\ &= 2\alpha(\eta(X)\Phi(Y, Z) + \eta(Z)\Phi(X, Y) + \eta(Y)\Phi(Z, X)). \end{aligned}$$

and applying this in (6.6), we obtain

$$\begin{aligned} \mathcal{B}(X, Y, Z) - \mathcal{B}(Y, X, Z) &= -(\nabla_Z \Phi)(\phi X, \phi Y) + (\nabla_Z \Phi)(X, Y) \\ &\quad - \eta(Y)g(\mathcal{A}\phi X, Z) + \eta(X)g(\mathcal{A}\phi Y, Z). \end{aligned}$$

By (6.3), the right hand side of this equality vanishes identically, so that $\mathcal{B}(X, Y, Z) - \mathcal{B}(Y, X, Z) = 0$, i.e. \mathcal{B} is symmetric with respect to X, Y .

Symmetrizing \mathcal{B} with respect to Y, Z , we find

$$\begin{aligned} \mathcal{B}(X, Y, Z) + \mathcal{B}(X, Z, Y) &= (\nabla_{\phi X} \Phi)(\phi Y, Z) + (\nabla_{\phi X} \Phi)(\phi Z, Y) \\ &\quad - \eta(Y)g(\mathcal{A}\phi X, Z) - \eta(Z)g(\mathcal{A}\phi X, Y). \end{aligned}$$

By the help of (6.2), we obtain $\mathcal{B}(X, Y, Z) + \mathcal{B}(X, Z, Y) = 0$, i.e. \mathcal{B} is antisymmetric with respect to Y, Z . The tensor \mathcal{B} having such symmetries must vanish identically, which implies (6.5). \square

Lemma 4. *For an almost α -paracosymplectic manifold, we also have*

$$(6.7) \quad (\nabla_{\phi X} \phi)Y - (\nabla_X \phi)\phi Y + \eta(Y)\mathcal{A}X - 2\alpha(g(X, Y)\xi - \eta(Y)X) = 0,$$

$$(6.8) \quad (\nabla_{\phi X} \phi)Y + \phi(\nabla_X \phi)Y - g(\mathcal{A}X, Y)\xi - 2\alpha(g(X, Y)\xi - \eta(Y)X) = 0.$$

Proof. Putting ϕY instead of Y in (6.5), we obtain

$$(6.9) \quad (\nabla_{\phi X} \phi)Y - \eta(Y)(\nabla_{\phi X} \phi)\xi - (\nabla_X \phi)\phi Y - 2\alpha(g(X, Y) - \eta(Y)\eta(X)\xi) = 0.$$

Using (3.10) and $(\nabla_{\phi X} \phi)\xi = \phi\mathcal{A}\phi X = -\mathcal{A}X - 2\alpha\phi^2 X$ in (6.9), we get (6.7). Equation (6.8) comes from (6.4) and (6.7). \square

Using (6.8), one can easily get following

Proposition 7. *For any almost α -paracosymplectic manifold, we have*

$$(6.10) \quad \phi(\nabla_{\phi X} \phi)Y + (\nabla_X \phi)Y = -2\alpha\eta(Y)\phi X + g(\alpha\phi X + hX, Y)\xi.$$

Theorem 2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -paracosymplectic manifold. Then, for any $X, Y \in \chi(M^{2n+1})$,*

$$\begin{aligned} (6.11) \quad R(X, Y)\xi &= d\alpha(X)(Y - \eta(Y)\xi) - d\alpha(Y)(X - \eta(X)\xi) + \alpha\eta(X)(\alpha Y + \phi hY) \\ &\quad - \alpha\eta(Y)(\alpha X + \phi hX) + (\nabla_X \phi h)Y - (\nabla_Y \phi h)X. \end{aligned}$$

Proof. We have the Ricci identity for the alteration the second covariant derivative $\nabla_{X,Y}\xi - \nabla_{Y,X}\xi = R(X, Y)\xi$. We notice that $\nabla_{X,Y}\xi = -(\nabla_X \mathcal{A})Y$. Now if we substitute \mathcal{A} according to (3.10) and applying the covariant derivative to the all summands in the result we obtain

$$(6.12) \quad \nabla_{X,Y}\xi = d\alpha(X)(Y - \eta(Y)\xi) + \alpha\eta(X)(\alpha Y + \phi hY) + (\nabla_X \phi h)Y.$$

\square

The identity for the curvature $R(X, Y)\xi$ greatly simplifies if $\dim(M) \geq 5$ according to the Proposition 1(vii).

Corollary 2. *For an almost α -paracosymplectic manifold M^{2n+1} , $n \geq 2$*

$$(6.13) \quad \begin{aligned} R(X, Y)\xi &= (f + \alpha^2)(\eta(X)Y - \eta(Y)X) + \alpha(\eta(X)\phi hY - \eta(Y)\phi hX) \\ &\quad + (\nabla_X \phi h)Y - (\nabla_Y \phi h)X, \end{aligned}$$

where $f = i_\xi d\alpha$.

7. ALMOST α -PARA-KENMOTSU MANIFOLDS

In this section we study particularly almost α -para-Kenmotsu manifolds if it is not otherwise stated.

Theorem 3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu manifold. Then, for any $X, Y \in \chi(M^{2n+1})$,*

$$(7.1) \quad R(X, Y)\xi = \alpha\eta(X)(\alpha Y + \phi hY) - \alpha\eta(Y)(\alpha X + \phi hX) + (\nabla_X \phi h)Y - (\nabla_Y \phi h)X.$$

Proof. It is direct consequence of the Theorem 2 for α is a constant. \square

Theorem 4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu manifold. Then, for any $X \in \chi(M^{2n+1})$ we have*

$$(7.2) \quad R(\xi, X)\xi = \alpha^2 \phi^2 X + 2\alpha \phi hX - h^2 X + \phi(\nabla_\xi h)X,$$

$$(7.3) \quad (\nabla_\xi h)X = -\alpha^2 \phi X - 2\alpha hX + \phi h^2 X - \phi R(X, \xi)\xi,$$

$$(7.4) \quad \frac{1}{2}(R(\xi, X)\xi + \phi R(\xi, \phi X)\xi) = \alpha^2 \phi^2 X - h^2 X.$$

$$(7.5) \quad S(X, \xi) = -2n\alpha^2 \eta(X) + g(\text{div}(\phi h), X)$$

$$(7.6) \quad S(\xi, \xi) = -2n\alpha^2 + \text{tr} h^2.$$

Proof. If we replace X by ξ and Y by X in (7.1) and use (3.10) we obtain (7.2). For the proof of (7.3), we apply the tensor field ϕ both sides of the (7.2) and recall $\nabla_\xi \phi = 0$. Hence we have

$$-\phi R(X, \xi)\xi = \alpha^2 \phi X + 2\alpha hX - \phi h^2 X + (\nabla_\xi h)X - g((\nabla_\xi h)X, \xi)\xi$$

Replacing X by ϕX in (7.2) we get

$$R(\xi, \phi X)\xi = \alpha^2 \phi^3 X + 2\alpha \phi h\phi X - h^2 \phi X + \phi(\nabla_\xi h)\phi X.$$

If we apply ϕ to the last equation we have

$$(7.7) \quad \phi R(\xi, \phi X)\xi = \alpha^2 \phi^2 X + 2\alpha h\phi X - h^2 X + (\nabla_\xi h)\phi X.$$

One can easily show that $\phi(\nabla_\xi h)X = -(\nabla_\xi h)\phi X$. Combining (7.2) with (7.7) we get (7.4).

Taking into account ϕ -basis and (7.1), Ricci curvature $S(X, \xi)$ can be given by

$$(7.8) \quad \begin{aligned} S(X, \xi) &= \sum_{i=1}^n [g(R(e_i, X)\xi, e_i) - g(R(\phi e_i, X)\xi, \phi e_i)] \\ &= -2n\alpha^2 \eta(X) - \sum_{i=1}^n (g((\nabla_X \phi h)e_i, e_i) - g((\nabla_X \phi h)\phi e_i, \phi e_i)) \\ &\quad + \sum_{i=1}^n (g((\nabla_{e_i} \phi h)X, e_i) - g((\nabla_{\phi e_i} \phi h)X, \phi e_i)) \end{aligned}$$

After some calculations we have

$$\sum_{i=1}^n (g((\nabla_X \phi h)e_i, e_i) - g((\nabla_X \phi h)\phi e_i, \phi e_i)) = 0,$$

$$\sum_{i=1}^n (g((\nabla_{e_i} \phi h)X, e_i) - g((\nabla_{\phi e_i} \phi h)X, \phi e_i)) = g(\operatorname{div}(\phi h), X).$$

Using the last two equations in (7.8) we obtain

$$S(X, \xi) = -2n\alpha^2\eta(X) + g(\operatorname{div}(\phi h), X).$$

By direct calculation, we find

$$S(\xi, \xi) = -2n\alpha^2 + \operatorname{tr} h^2.$$

□

Proposition 8. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu manifold. Then, for any $X, Y, Z \in \chi(M^{2n+1})$ we have*

$$\begin{aligned} & g(R(\xi, X)Y, Z) + g(R(\xi, X)\phi Y, \phi Z) - g(R(\xi, \phi X)\phi Y, Z) - g(R(\xi, \phi X)Y, \phi Z) \\ &= 2(\nabla_{hX}\Phi)(Y, Z) + 2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) \\ (7.9) \quad & -2\alpha\eta(Z)g(\phi hX, Y) + 2\alpha\eta(Y)g(\phi hX, Z). \end{aligned}$$

Proof. The symmetries of the curvature tensor give $g(R(\xi, X)Y, Z) = g(X, R(Y, Z)\xi)$ and then, using (7.1), the left hand side can be written as

$$(7.10) \quad 2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) + \mathcal{F}(X, Y, Z) - \mathcal{F}(X, Z, Y),$$

where

$$\begin{aligned} \mathcal{F}(X, Y, Z) &= g(X, (\nabla_Y \phi h)Z) + \phi(\nabla_Y \phi h)\phi Z \\ &\quad + g(X, (\nabla_{\phi Y} \phi h)\phi Z) - g(\phi X, (\nabla_{\phi Y} \phi h)Z). \end{aligned}$$

By direct computations we have

$$(7.11) \quad \phi(\nabla_Y \phi h)\phi Z + (\nabla_Y \phi h)Z = (\nabla_Y \phi)hZ - h(\nabla_Y \phi)Z,$$

and

$$\begin{aligned} g(X, (\nabla_{\phi Y} \phi h)\phi Z) - g(\phi X, (\nabla_{\phi Y} \phi h)Z) &= -g(\phi X, \phi((\nabla_{\phi Y} \phi h)\phi Z)) \\ (7.12) \quad &\quad + \eta(X)\eta((\nabla_{\phi Y} \phi h)\phi Z) - g(\phi X, (\nabla_{\phi Y} \phi h)Z). \end{aligned}$$

Using (7.11), (7.12), (6.10) and the equality $\eta((\nabla_{\phi Y} \phi h)\phi Z) = g(hZ, \alpha\phi Y - hY)$, we obtain

$$(7.13) \quad \mathcal{F}(X, Y, Z) = -2g(hX, (\nabla_Y \phi)Z) - 2\alpha\eta(Z)g(h\phi Y, X) + 2\alpha\eta(X)g(h\phi Y, Z).$$

Using (7.13) in (7.10), the required formula $\sigma_{Y, Z, hX}(\nabla_Y \Phi)(Z, hX) = d\Phi(Y, Z, hX)$ and $d\Phi = 2\alpha\eta \wedge \Phi$. □

Theorem 5. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu manifold with para-Kaehler leaves. Then the following identity holds*

$$(7.14) \quad Q\phi - \phi Q = l\phi - \phi l - 4\alpha(1-n)h - \eta \otimes \phi Q + (\eta \circ Q\phi) \otimes \xi,$$

where l denotes the Jacobi operator, defined by $lX = R(X, \xi)\xi$.

Proof. We recall the formula (7.1)

$$R(X, Y)\xi = \alpha\eta(X)(\alpha Y + \phi hY) - \alpha\eta(Y)(\alpha X + \phi hX) + (\nabla_X \phi h)Y - (\nabla_Y \phi h)X.$$

On the other hand

$$(7.15) \quad (\nabla_X \phi h)Y = (\nabla_X \phi)hY + \phi((\nabla_X h)Y).$$

Using (5.2) and (7.15), we obtain

$$(7.16) \quad \begin{aligned} R(X, Y)\xi &= \alpha\eta(X)(\alpha Y + \phi hY) - \alpha\eta(Y)(\alpha X + \phi hX) \\ &\quad + \phi((\nabla_X h)Y - (\nabla_Y h)X) + (\nabla_X \phi)hY - (\nabla_Y \phi)hX. \end{aligned}$$

By assumption M^{2n+1} has para-Kaehler leaves thus by (5.2) $(\nabla_X \phi)hY = \alpha g(\phi X, hY)\xi + g(hX, hY)\xi$ in consequence, as $h\phi$ is symmetric, $(\nabla_X \phi)hY - (\nabla_Y \phi)hX$ vanishes identically. Since h is a symmetric operator we easily get

$$(7.17) \quad g((\nabla_X h)Y - (\nabla_Y h)X, \xi) = g((\nabla_X h)\xi, Y) - g((\nabla_Y h)\xi, X).$$

Using the formulas (3.10), $h\xi = 0$ and $\phi h + h\phi = 0$ in (7.17) we find

$$(7.18) \quad g((\nabla_X h)Y - (\nabla_Y h)X, \xi) = 2g(\phi h^2 X, Y).$$

By applying ϕ to (7.16) and using $\phi^2 = I - \eta \otimes \xi$ and (7.18) we obtain

$$(7.19) \quad \begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X &= \phi R(X, Y)\xi + 2g(\phi h^2 X, Y)\xi \\ &\quad - \alpha^2(\eta(X)\phi Y - \eta(Y)\phi X) - \alpha(\eta(X)hY - \eta(Y)hX). \end{aligned}$$

Now we suppose that P is a fixed point of M and X, Y, Z are vector fields such that $(\nabla X)_P = (\nabla Y)_P = (\nabla Z)_P = 0$. The Ricci identity for ϕ

$$R(X, Y)\phi Z - \phi R(X, Y)Z = (\nabla_X \nabla_Y \phi)Z - (\nabla_Y \nabla_X \phi)Z - (\nabla_{[X, Y]}\phi)Z,$$

at the point P , reduces to the form

$$R(X, Y)\phi Z - \phi R(X, Y)Z = \nabla_X(\nabla_Y \phi)Z - \nabla_Y(\nabla_X \phi)Z.$$

Due to our assumption that M^{2n+1} has para-Kaehler leaves from (5.2) we obtain at P By virtue of the integrability condition we have, at P ,

$$(7.20) \quad \begin{aligned} R(X, Y)\phi Z - \phi R(X, Y)Z &= \nabla_X(\nabla_Y \phi)Z - \nabla_Y(\nabla_X \phi)Z \\ &= \alpha(g((\nabla_X \phi)Y - (\nabla_Y \phi)X, Z)\xi - \eta(Z)((\nabla_X \phi)Y - (\nabla_Y \phi)X)) \\ &\quad + g((\nabla_X h)Y - (\nabla_Y h)X, Z)\xi - \eta(Z)((\nabla_X h)Y - (\nabla_Y h)X)) \\ &\quad + g(\alpha\phi Y + hY, Z)(\alpha\phi^2 X + \phi hX) - g(\alpha\phi X + hX, Z)(\alpha\phi^2 Y + \phi hY) \\ &\quad - g(Z, \alpha\phi^2 X + \phi hX)(\alpha\phi Y + hY) + g(Z, \alpha\phi^2 Y + \phi hY)(\alpha\phi X + hX). \end{aligned}$$

Using (5.2) and (7.19) in (7.20) we find

$$(7.21) \quad \begin{aligned} R(X, Y)\phi Z - \phi R(X, Y)Z &= g(\phi R(X, Y)\xi, Z)\xi - \eta(Z)\phi R(X, Y)\xi \\ &\quad + g(\alpha\phi Y + hY, Z)(\alpha\phi^2 X + \phi hX) - g(\alpha\phi X + hX, Z)(\alpha\phi^2 Y + \phi hY) \\ &\quad - g(Z, \alpha\phi^2 X + \phi hX)(\alpha\phi Y + hY) + g(Z, \alpha\phi^2 Y + \phi hY)(\alpha\phi X + hX). \end{aligned}$$

Using (2.1) and the curvature tensor properties we get

$$(7.22) \quad g(\phi R(\phi X, \phi Y)Z, \phi W) = -g(R(Z, W)\phi X, \phi Y) + \eta(R(\phi X, \phi Y)Z)\eta(W).$$

Then by (7.21) and (7.22) we obtain

$$\begin{aligned}
 g(\phi R(\phi X, \phi Y)Z, \phi W) &= -g(\phi R(Z, W)X, \phi Y) + \eta(R(\phi X, \phi Y)Z)\eta(W) \\
 &\quad + \eta(X)g(\phi R(Z, W)\xi, \phi Y) \\
 &\quad - g(\alpha\phi W + hW, X)(g(\alpha\phi^2 Z, \phi Y) + g(\phi hZ, \phi Y)) \\
 &\quad + g(\alpha\phi Z + hZ, X)(g(\alpha\phi^2 W, \phi Y) + g(\phi hW, \phi Y)) \\
 &\quad + g(X, \alpha\phi^2 Z + \phi hZ)((g(\alpha\phi W, \phi Y) + g(hW, \phi Y)) \\
 &\quad - g(X, \alpha\phi^2 W + \phi hW)((g(\alpha\phi Z, \phi Y) + g(hZ, \phi Y)).
 \end{aligned}
 \tag{7.23}$$

Replacing in (7.21) X, Y by $\phi X, \phi Y$ respectively, and taking the inner product with ϕW , we get

$$\begin{aligned}
 g(R(\phi X, \phi Y)\phi Z, \phi W) - g(\phi R(\phi X, \phi Y)Z, \phi W) &= -\eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W) \\
 &\quad + g(\alpha\phi^2 Y + h\phi Y, Z)g(\alpha\phi^3 X + \phi h\phi X, \phi W) \\
 &\quad - g(\alpha\phi^2 X + h\phi X, Z)g(\alpha\phi^3 Y + \phi h\phi Y, \phi W) \\
 &\quad - g(Z, \alpha\phi^3 X + \phi h\phi X)g(\alpha\phi^2 Y + h\phi Y, \phi W) \\
 &\quad + g(Z, \alpha\phi^3 Y + \phi h\phi Y)g(\alpha\phi^2 X + h\phi X, \phi W).
 \end{aligned}
 \tag{7.24}$$

Comparing (7.23) to (7.24) we get by direct computation

$$\begin{aligned}
 g(R(\phi X, \phi Y)\phi Z, \phi W) &= g(R(Z, W)X, Y) - \eta(R(Z, W)X)\eta(Y) \\
 &\quad - \eta(X)g(R(Z, W)\xi, Y) + \eta(R(\phi X, \phi Y)Z)\eta(W) \\
 &\quad - \eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W) \\
 &\quad - 2\alpha g(X, Z)g(Y, \phi hW) + 2\alpha\eta(X)\eta(Z)g(\phi hW, Y) \\
 &\quad + 2\alpha g(Y, Z)g(X, \phi hW) - 2\alpha\eta(Y)\eta(Z)g(\phi hW, X) \\
 &\quad + 2\alpha g(X, W)g(Y, \phi hZ) - 2\alpha\eta(X)\eta(W)g(\phi hZ, Y) \\
 &\quad + 2\alpha\eta(Y)\eta(W)g(X, \phi hZ) - 2\alpha g(Y, W)g(\phi hZ, X).
 \end{aligned}
 \tag{7.25}$$

Let $\{e_i, \phi e_i, \xi\}, i \in \{1, \dots, n\}$, be a local ϕ -basis. Setting $Y = Z = e_i$ in (7.25), we have

$$\begin{aligned}
 \sum_{i=1}^n g(R(\phi X, \phi e_i)\phi e_i, \phi W) &= \sum_{i=1}^n (g(R(e_i, W)X, e_i) - \eta(X)g(R(e_i, W)\xi, e_i) + \eta(R(\phi X, \phi e_i)e_i)\eta(W) \\
 &\quad - 2\alpha g(X, e_i)g(e_i, \phi hW) + 2\alpha g(e_i, e_i)g(\phi hW, X) \\
 &\quad + 2\alpha g(X, W)g(e_i, \phi h e_i) - 2\alpha\eta(X)\eta(W)g(\phi h e_i, e_i) \\
 &\quad - 2\alpha g(e_i, W)g(\phi h e_i, X)).
 \end{aligned}
 \tag{7.26}$$

On the other hand, putting $Y = Z = \phi e_i$ in (7.25), we get

$$\begin{aligned}
 \sum_{i=1}^n g(R(\phi X, e_i)e_i, \phi W) &= \sum_{i=1}^n (g(R(\phi e_i, W)X, \phi e_i) - \eta(X)g(R(\phi e_i, W)\xi, \phi e_i) + \eta(R(\phi X, e_i)\phi e_i)\eta(W) \\
 &\quad - 2\alpha g(X, \phi e_i)g(\phi e_i, \phi hW) + 2\alpha g(\phi e_i, \phi e_i)g(\phi hW, X) \\
 &\quad + 2\alpha g(X, W)g(\phi e_i, \phi h\phi e_i) - 2\alpha\eta(X)\eta(W)g(\phi h\phi e_i, \phi e_i) \\
 &\quad - 2\alpha g(\phi e_i, W)g(\phi h\phi e_i, X)).
 \end{aligned}
 \tag{7.27}$$

Using the definition of the Ricci operator, (7.26) and (7.27), one can easily get

$$(7.28) \quad \begin{aligned} -\phi Q\phi X + \phi l\phi X + QX - lX &= \eta(X)Q\xi + 4\alpha(1-n)\phi hX \\ &+ \sum_{i=1}^n (g(R(\phi X, e_i)\phi e_i, \xi) - g(R(\phi X, \phi e_i)e_i, \xi))\xi. \end{aligned}$$

Finally, applying ϕ to (7.28) and using $\phi^2 = I - \eta \otimes \xi$, we obtain the requested equation. \square

Theorem 6. *Let M^{2n+1} be an almost α -para-Kenmotsu manifold of constant sectional curvature c . Then $c = -\alpha^2$ and $h^2 = 0$.*

Proof. If an almost α -para-Kenmotsu manifold of constant sectional curvature c then

$$(7.29) \quad R(\xi, X)\xi = c(\eta(X)\xi - X) = \phi R(\xi, \phi X)\xi.$$

for any $X \in \Gamma(M)$. Using this relation in (7.4) we have

$$(7.30) \quad h^2 X = (\alpha^2 + c)\phi^2 X$$

Differentiating (7.30) with respect to ξ and using $\nabla_\xi \phi = 0$ we find $\nabla_\xi h^2 = 0$ which implies

$$(\nabla_\xi h) \circ h + h \circ (\nabla_\xi h) = 0.$$

Applying ∇_ξ to the above equation and using (7.3), we get $(\nabla_\xi h)^2 = 0$. Since $\nabla_\xi h$ is symmetric operator one easily have

$$(7.31) \quad 0 = g((\nabla_\xi h)^2 X, Y) = g((\nabla_\xi h)X, (\nabla_\xi h)Y).$$

By virtue of (7.30), (7.29) and (7.3) we find

$$(\nabla_\xi h)X = -2\alpha hX$$

Hence (7.31) is reduce to $4\alpha^2 g(h^2 X, Y) = 4\alpha^2(\alpha^2 + c)g(\phi X, \phi Y) = 0$.

Because of $\alpha \neq 0$, we obtain $c = -\alpha^2$ and $h^2 = 0$. \square

The proof of the following theorem is exactly same with almost Kenmotsu manifolds [19], therefore we omit their proofs.

Theorem 7. *Let M^{2n+1} be an almost α - para Kenmotsu manifold with $h = 0$. Then M^{2n+1} is a locally warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para Kaehler manifold, M_1 is an open interval with coordinate t , and $f^2 = we^{2t}$ for some positive constant.*

Remark 2. *Almost Kenmotsu manifolds in almost contact metric geometry appeared in [19], [24] and [29]. These manifolds were extensively studied e.g. [19], [20], [33], [37]. Arbitrary almost Kenmotsu manifold can be locally deformed conformally to almost cosymplectic manifold. Almost Kenmotsu manifolds were generalized to almost α -Kenmotsu, $\alpha = \text{const}$, and subsequently to almost α -cosymplectic manifolds.*

8. HARMONIC VECTOR FIELDS

Let (M, g) be smooth, oriented, connected pseudo-Riemannian manifold and (TM, g^S) its tangent bundle endowed with the Sasaki metric (also referred to as Kaluza-Klein metric in Mathematical Physics) g^S . By definition, the *energy* of a smooth vector field

V on M is the energy corresponding $V : (M, g) \rightarrow (TM, g^s)$. When M is compact, the *energy* of V is determined by

$$E(V) = \frac{1}{2} \int_M (tr_g V^* g^s) dv = \frac{n}{2} vol(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv.$$

The non-compact case, one can take into account over relatively compact domains. It can be shown that $V : (M, g) \rightarrow (TM, g^s)$ is harmonic map if and only if

$$(8.1) \quad tr [R(\nabla.V, V).] = 0, \quad \nabla^* \nabla V = 0,$$

where

$$(8.2) \quad \nabla^* \nabla V = - \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V)$$

is the rough Laplacian with respect to a pseudo-orthonormal local frame $\{e_1, \dots, e_n\}$ on (M, g) with $g(e_i, e_i) = \varepsilon_i = \pm 1$ for all indices $i = 1, \dots, n$.

If (M, g) is a compact Riemannian manifold, only parallel vector fields define harmonic maps.

Next, for any real constant $\rho \neq 0$, let $\chi^\rho(M) = \{W \in \chi(M) : \|W\|^2 = \rho\}$. We consider vector fields $V \in \chi^\rho(M)$ which are critical points for the energy functional $E|_{\chi^\rho(M)}$, restricted to vector fields of the same length. The Euler-Lagrange equations of this variational condition yield that V is a harmonic vector field if and only if

$$(8.3) \quad \nabla^* \nabla V \text{ is collinear to } V.$$

This characterization is well known in the Riemannian case ([3, 23, 25]). Using same arguments in pseudo-Riemannian case, G. Calvaruso [7] proved that same result is still valid for vector fields of constant length, if it is not lightlike.

Let $T_1 M$ denote the unit tangent sphere bundle over M , and again by g^s the metric induced on $T_1 M$ by the Sasaki metric of TM . Then, it is shown that in [1], the map on $V : (M, g) \rightarrow (T_1 M, g^s)$ is harmonic if V is a harmonic vector field and the additonal condition

$$(8.4) \quad tr[R(\nabla.V, V).] = 0$$

is satisfied. G. Calvaruso [7] also investigated harmonicity properties for left-invariant vector fields on three-dimensional Lorentzian Lie groups, obtaining several classification results and new examples of critical points of energy functionals.

In the non-compact case, conditions (8.1) and (8.3) are respectively taken as definitions of harmonic vector fields and of vector fields defining harmonic maps.

Recently, D. Perrone proved that the characteristic vector field of an almost cosymplectic three-manifold is minimal if and only if it is an eigenvector of the Ricci operator.

Theorem 8. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu manifold. Then*

$$\bar{\Delta} \xi = -\nabla^* \nabla \xi = (2n\alpha^2 - tr(h^2))\xi - Q\xi_{|\ker \eta}.$$

Proof. Now, let $(e_i, \phi e_i, \xi), i = 1, \dots, n$, be a local orthogonal ϕ -basis. Then we obtain

$$\begin{aligned} \bar{\Delta}\xi &= -\sum_{i=1}^n (\nabla_{e_i} \nabla_{e_i} \xi - \nabla_{\nabla_{e_i} e_i} \xi - \nabla_{\phi e_i} \nabla_{\phi e_i} \xi + \nabla_{\nabla_{\phi e_i} \phi e_i} \xi) \\ &= -\sum_{i=1}^n ((\nabla_{e_i} \nabla \xi) e_i - (\nabla_{\phi e_i} \nabla \xi) \phi e_i) \\ &\stackrel{(3.10)}{=} -\sum_{i=1}^n ((\nabla_{e_i} A) e_i - (\nabla_{\phi e_i} A) \phi e_i) \\ &= -\operatorname{div} \phi h + 2n\alpha^2 \xi \end{aligned}$$

By (7.5) and (7.6) we get

$$\bar{\Delta}\xi = (2n\alpha^2 - \operatorname{tr}(h^2))\xi - Q\xi|_{\ker \eta}.$$

This ends the proof. \square

As an immediate consequence of Theorem 8 we obtain following theorem.

Theorem 9. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu manifold. ξ is a harmonic vector field if and only if the characteristic vector field is an eigenvector of the Ricci operator.*

9. CONFORMAL AND D -HOMOTHETIC DEFORMATIONS.

Let M^{2n+1} be an almost α -paracosymplectic manifold and (ϕ, ξ, η, g) be almost α -paracosymplectic structure. Let $\mathcal{R}_\eta(M^{2n+1})$ be the set of the locally defined smooth functions f on M^{2n+1} such that $df \wedge \eta = 0$, whenever df is defined.

Let M^{2n+1} be an almost paracontact metric manifold. Let f be a function on M^{2n+1} , $f > 0$ everywhere. Consider a deformation of the structure

$$(9.1) \quad \phi \mapsto \phi' = \phi, \quad \xi \mapsto \xi' = \frac{1}{f}\xi, \quad \eta \mapsto \eta' = f\eta, \quad g \mapsto g' = fg,$$

we call (ϕ', ξ', η', g') for rather obvious reasons conformal deformation of (ϕ, ξ, η, g) . Respectively we say that almost paracontact metric manifold $(M^{2n+1}, \phi', \xi', \eta')$ is conformal to $(M^{2n+1}, \phi, \xi, \eta, g)$. Almost paracontact metric manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$ and $(M^{2n+1}, \phi', \xi', \eta', g')$ are called locally conformal if there is an open covering $(U_i)_{i \in I}$, $M^{2n+1} = \bigcup U_i$, such that almost paracontact metric manifolds $(U_i, \phi|_{U_i}, \xi|_{U_i}, \eta|_{U_i}, g|_{U_i})$ and $(U_i, \phi'|_{U_i}, \xi'|_{U_i}, \eta'|_{U_i}, g'|_{U_i})$ are conformal.

Theorem 10. *Arbitrary almost α -paracosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 2$ is locally conformal to an almost paracosymplectic manifold. In the other words near each point $p \in M^{2n+1}$ there is defined function u , such that a structure (ϕ', ξ', η', g')*

$$(9.2) \quad \phi' = \phi, \quad \xi' = e^{2u}\xi, \quad \eta' = e^{-2u}\eta, \quad g' = e^{-2u}g,$$

is almost paracosymplectic. The function u is unique up to additive constant and $\alpha\eta = du$.

Proof. Let u be a local function defined near a given point $p \in M^{2n+1}$. Then we directly verify that the fundamental form of the structure (ϕ', ξ', η', g') is closed if and only if $du = \alpha\eta$. Indeed $\Phi'(X, Y) = g'(\phi'X, Y) = e^{-2u}g(\phi X, Y) = e^{-2u}\Phi(X, Y)$ and

$$(9.3) \quad d\Phi' = -2e^{-2u}du \wedge \Phi + e^{-2u}d\Phi = 2e^{-2u}(-du + \alpha\eta) \wedge \Phi,$$

by the Lemma 1 $d\Phi'$ vanishes everywhere only if the 1-form $-du + \alpha\eta$ is identically zero. Notice that in the case when dimension of M^{2n+1} is ≥ 5 , such function always locally

exists for according to the Proposition 1(vii) the 1-form $\beta = \alpha\eta$ is closed everywhere $d\beta = d\alpha \wedge \eta = f\eta \wedge \eta = 0$, so from the Poincare lemma we have a local solution. Similarly we obtain $d\eta' = -2e^{-2u}du \wedge \eta = -2e^{-2u}\alpha\eta \wedge \eta = 0$. Thus let U be an open set, $p \in U$, such that the function u is defined on U , then the manifold $(U, \phi', \xi', \eta', g')$ is almost para-cosymplectic. \square

Consider a $D_{\gamma, \beta}$ -homothetic deformation of (ϕ, ξ, η, g) into an almost paracontact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defined as

$$(9.4) \quad \tilde{\phi} = \phi, \quad \tilde{\xi} = \frac{1}{\beta}\xi, \quad \tilde{\eta} = \beta\eta, \quad \tilde{g} = \gamma g + (\beta^2 - \gamma)\eta \otimes \eta,$$

where γ is positive constant and $\beta \in \mathcal{R}_\eta(M^{2n+1})$, $\beta \neq 0$ at any point of M^{2n+1} . Since $d\beta \wedge \eta = 0$, it follows that

$$d\tilde{\eta} = d\beta \wedge \eta + \beta d\eta = 0,$$

and moreover $d\tilde{\Phi} = 2(\frac{\gamma}{\beta})\tilde{\eta} \wedge \tilde{\Phi}$, since fundamental two forms $\Phi, \tilde{\Phi}$ are related by $\tilde{\Phi} = \gamma\Phi$. So, deformed structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ can be written as

$$\tilde{\Phi} = \gamma\Phi, \quad d\tilde{\eta} = 0, \quad d\tilde{\Phi} = 2\frac{\gamma}{\beta}\tilde{\eta} \wedge \tilde{\Phi},$$

for $d\beta = d\beta(\xi)\eta$ and $\frac{\gamma}{\beta} \in \mathcal{R}_\eta(M^{2n+1})$.

Thus a $D_{\gamma, \beta}$ -homothetic deformation of an almost α -paracosymplectic structure (ϕ, ξ, η, g) gives a new almost $(\frac{\gamma}{\beta})$ -paracosymplectic structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on the same manifold.

Following the definition of locally conformal almost paracontact metric manifolds we define the notion of locally $D_{\gamma, \beta}$ -homothetic almost α -paracosymplectic manifolds. By the Proposition 1(vii) we have the following

Theorem 11. *An almost α -paracosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 2$ is locally $D_{\gamma, \alpha}$ -homothetic to an almost para-Kenmotsu manifold on the set $U : \alpha \neq 0$.*

Proposition 9. *Let $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be an almost α -paracosymplectic structure obtained from (ϕ, ξ, η, g) by a $D_{\gamma, \beta}$ -homothetic deformation. Then we have the following relationship between the Levi-Civita connections $\tilde{\nabla}$ and ∇ .*

$$(9.5) \quad \tilde{\nabla}_X Y = \nabla_X Y - \left(\frac{\beta^2 - \gamma}{\beta^2} \right) g(\mathcal{A}X, Y)\xi + \frac{d\beta(\xi)}{\beta} \eta(Y)\eta(X)\xi.$$

Proof. By Kozsul's formula we have

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(X, Z) - Z\tilde{g}(X, Y) \\ &\quad + \tilde{g}([X, Y], Z) + \tilde{g}([Z, X], Y) + \tilde{g}([Z, Y], X), \end{aligned}$$

for any vector fields X, Y, Z . By using $\tilde{g} = \gamma g + (\beta^2 - \gamma)\eta \otimes \eta$ in the last equation we obtain

$$(9.6) \quad \begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= 2\gamma g(\nabla_X Y, Z) + 2\beta d\beta(\xi)\eta(X)\eta(Y)\eta(Z) \\ &\quad + 2(\beta^2 - \gamma)[\eta(\nabla_X Y)\eta(Z) + g(Y, \nabla_X \xi)\eta(Z)]. \end{aligned}$$

Moreover we have

$$(9.7) \quad 2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2\gamma g(\nabla_X Y, Z) + 2(\beta^2 - \gamma)\eta(\tilde{\nabla}_X Y)\eta(Z)$$

and

$$(9.8) \quad \eta(\tilde{\nabla}_X Y) = \frac{1}{\beta^2}\tilde{g}(\tilde{\nabla}_X Y, \xi).$$

Using (9.7) and (9.8) in (9.6) we find

$$(9.9) \quad \gamma g(\tilde{\nabla}_X Y, Z) + \frac{(\beta^2 - \gamma)}{\beta^2} \tilde{g}(\tilde{\nabla}_X Y, \xi) \eta(Z) = \gamma g(\nabla_X Y, Z) + \beta d\beta(\xi) \eta(X) \eta(Y) \eta(Z) + (\beta^2 - \gamma) [\eta(\nabla_X Y) \eta(Z) + g(Y, \nabla_X \xi) \eta(Z)].$$

Setting $Z = \xi$ in (9.6) we get

$$(9.10) \quad \tilde{g}(\tilde{\nabla}_X Y, \xi) = \gamma g(\nabla_X Y, \xi) + \beta d\beta(\xi) \eta(X) \eta(Y) + (\beta^2 - \gamma) [\eta(\nabla_X Y) + g(Y, \nabla_X \xi)].$$

Using (9.10) in (9.9), by a direct computation we have (9.5). \square

Proposition 10. *Let $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be an almost α -paracosymplectic structure obtained from (ϕ, ξ, η, g) by a $D_{\gamma, \beta}$ -homothetic deformation. Then the following identities hold:*

$$(9.11) \quad \mathcal{A}' X = \frac{1}{\beta} \mathcal{A} X,$$

$$(9.12) \quad \tilde{h} X = \frac{1}{\beta} h X,$$

$$(9.13) \quad \tilde{R}(X, Y) \tilde{\xi} = \frac{1}{\beta} R(X, Y) \xi + \frac{1}{\beta^2} d\beta(\xi) [\eta(X) \mathcal{A} Y - \eta(Y) \mathcal{A} X],$$

for any vector fields X, Y, Z .

By using (3.10), (9.4) and (9.5) we obtain

$$\mathcal{A}' X = \frac{X(\beta)}{\beta^2} \xi - \frac{1}{\beta} \nabla_X \xi - \frac{1}{\beta^2} d\beta(\xi) \eta(X) \xi.$$

By virtue of the definition β , the last equation reduces to (9.11). (9.12) follows from (9.4) by using the properties of h . First, from (9.4) and (9.5) we have

$$(9.14) \quad \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{\xi} = \nabla_X \tilde{\nabla}_Y \tilde{\xi} - \frac{(\beta^2 - \gamma)}{\beta^2} g(\mathcal{A} X, \tilde{\nabla}_Y \tilde{\xi}) \xi + \frac{1}{\beta} d\beta(\xi) \eta(X) \eta(\tilde{\nabla}_Y \tilde{\xi}) \xi,$$

$$(9.15) \quad \tilde{\nabla}_Y \tilde{\xi} = \frac{1}{\beta} \nabla_Y \xi.$$

Using the properties of \mathcal{A} and (9.15), (9.14) reduces to

$$(9.16) \quad \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{\xi} = \frac{X(\beta)}{\beta^2} \mathcal{A} Y + \frac{1}{\beta} \nabla_X \nabla_Y \xi + \frac{(\beta^2 - \gamma)}{\beta^3} g(\mathcal{A} X, \mathcal{A} Y) \xi.$$

Then by (9.15) and (9.16) we obtain by a straightforward calculation

$$\begin{aligned} \tilde{R}(X, Y) \tilde{\xi} &= \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{\xi} - \tilde{\nabla}_Y \tilde{\nabla}_X \tilde{\xi} - \tilde{\nabla}_{[X, Y]} \tilde{\xi} \\ &= \frac{X(\beta)}{\beta^2} \mathcal{A} Y - \frac{Y(\beta)}{\beta^2} \mathcal{A} X + \frac{1}{\beta} R(X, Y) \xi \end{aligned}$$

which gives (9.13).

10. ALMOST α -PARA-KENMOTSU (κ, μ, ν) -SPACES

In this section we study almost α -para-Kenmotsu manifolds under assumption that the curvature $R(X, Y)Z$ satisfies so called (κ, μ, ν) -nullity condition, i.e.

$$(10.1) \quad R(X, Y)\xi = \eta(Y)BX - \eta(X)BY,$$

where B is a $(1, 1)$ -tensor field defined by

$$(10.2) \quad BX = \kappa\phi^2X + \mu hX + \nu\phi hX$$

for $\kappa, \mu, \nu \in \mathcal{R}_\eta(M^{2n+1})$. Particularly $B\xi = 0$.

If an almost α -paracosymplectic manifold satisfies (10.1), then the manifold is said to be almost α -paracosymplectic (κ, μ, ν) -space and (ϕ, ξ, η, g) be called almost α -paracosymplectic (κ, μ, ν) -structure.

Using (9.13) and after some calculations one can prove following proposition.

Proposition 11. *Under the same assumptions of Proposition 10, if (ϕ, ξ, η, g) is an almost α -para-Kenmotsu (κ, μ, ν) -structure then $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost $(\frac{\gamma}{\beta})$ -paracosymplectic $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$ structure, where*

$$\tilde{\kappa} = \frac{\kappa}{\beta^2} + \frac{\alpha}{\beta^3}d\beta(\xi), \quad \tilde{\mu} = \frac{\mu}{\beta}, \quad \tilde{\nu} = \frac{\nu}{\beta} + \frac{d\beta(\xi)}{\beta^2}; \quad \tilde{\kappa}, \tilde{\mu}, \tilde{\nu} \in \mathcal{R}_{\tilde{\eta}}(M^{2n+1}).$$

For an almost α -paracosymplectic (κ, μ, ν) -space we may consider a scalar invariant with respect to the $D_{\gamma, \beta}$ -homothety, that is a function $I(\alpha, \kappa, \mu, \nu)$ with the property that $I(\alpha, \kappa, \mu, \nu) = I(\alpha', \kappa', \mu', \nu')$ for arbitrary $D_{\gamma, \beta}$ -homothety. In the case $\mu \neq 0$ by direct computations we find that

$$(10.3) \quad I_0(\alpha, \kappa, \mu, \nu) = \frac{\kappa - \alpha\nu}{\mu^2},$$

is an invariant. An almost α -paracosymplectic space will be called of constant I_0 -type if $\mu \neq 0$ and $I_0 = \text{const}$ is a constant.

Proposition 12. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu (κ, μ, ν) -space. Then the following identities hold:*

$$(10.4) \quad l = \kappa\phi^2 + \mu h + \nu\phi h,$$

$$(10.5) \quad l\phi - \phi l = 2\mu h\phi - 2\nu h,$$

$$(10.6) \quad h^2 = (\kappa + \alpha^2)\phi^2,$$

$$(10.7) \quad \nabla_\xi h = -(2\alpha + \nu)h + \mu h\phi,$$

$$(10.8) \quad \nabla_\xi h^2 = -2(2\alpha + \nu)(\kappa + \alpha^2)\phi^2,$$

$$(10.9) \quad \xi(\kappa) = -2(2\alpha + \nu)(\kappa + \alpha^2),$$

$$(10.10) \quad \begin{aligned} R(\xi, X)Y &= \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(X, hY)\xi - \eta(Y)hX) \\ &\quad + \nu(g(X, \phi hY)\xi - \eta(Y)\phi hX), \end{aligned}$$

$$(10.11) \quad Q\xi = 2n\kappa\xi,$$

$$(10.12) \quad (\nabla_X \phi)Y = g(Y, hX + \alpha\phi X)\xi - \eta(Y)(hX + \alpha\phi X),$$

$$(10.13) \quad (\nabla_X \phi h)Y - (\nabla_Y \phi h)X = (\kappa + \alpha^2)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + (\nu + \alpha)(\eta(Y)\phi hX - \eta(X)\phi hY),$$

$$(10.14) \quad (\nabla_X h)Y - (\nabla_Y h)X = (\kappa + \alpha^2)(\eta(Y)\phi X - \eta(X)\phi Y + 2g(Y, \phi X)\xi) + \mu(\eta(Y)\phi hX - \eta(X)\phi hY) + (\nu + \alpha)(\eta(Y)hX - \eta(X)hY),$$

for all vector fields X, Y on M^{2n+1} .

Proof. From (10.1) we get

$$(10.15) \quad lX = R(X, \xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX + \nu \phi hX$$

which gives (10.4). By replacing X by ϕX in (10.15), we have

$$l\phi X = \kappa\phi X + \mu h\phi X - \nu hX.$$

By applying now ϕ to (10.15), we obtain

$$\phi lX = \kappa\phi X - \mu h\phi X + \nu hX.$$

(10.5) comes from the last two equations. From (10.15) we easily get

$$(10.16) \quad \phi l\phi X = \kappa\phi^2 X - \mu hX - \nu \phi hX.$$

Then by (10.15) and (10.16) we obtain

$$-lX - \phi l\phi X = 2(\alpha^2 \phi^2 X - h^2 X).$$

Comparing this equation with (7.4), we have (10.6). (10.7) can be easily get from using (10.6) in (7.3). From (10.6) we find

$$\nabla_\xi h^2 = (\nabla_\xi h)h + h(\nabla_\xi h) = -2(2\alpha + \nu)h^2.$$

(10.8) comes from using (10.6) in the last equation. One can easily get (10.9) by differentiating (10.6) along ξ . Using $g(R(\xi, X)Y, Z) = g(R(Y, Z)\xi, X)$ and (10.1) we have

$$(10.17) \quad g(R(Y, Z)\xi, X) = \kappa(\eta(Z)g(X, Y) - \eta(Y)g(X, Z)) + \mu(\eta(Z)g(X, hY) - \eta(Y)g(X, hZ)) + \nu(\eta(Z)g(X, \phi hY) - \eta(Y)g(X, \phi hZ)).$$

The last equation completes the proof of (10.10). Then the definition of the Ricci operator directly gives (10.11). For (10.12), by virtue of (10.17), the left hand side of Eq. (7.9) can be written as

$$2\kappa(\eta(Z)g(X, Y) - \eta(Y)g(X, Z)).$$

So, (7.9) reduces to

$$(10.18) \quad 2\kappa(\eta(Z)g(X, Y) - \eta(Y)g(X, Z)) = 2(\nabla_{hX}\Phi)(Y, Z) + 2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) - 2\alpha\eta(Z)g(\phi hX, Y) + 2\alpha\eta(Y)g(\phi hX, Z).$$

From the last equation we have

$$(\nabla_{hX}\Phi)(Y, Z) = (\kappa + \alpha^2)(\eta(Z)g(X, Y) - \eta(Y)g(X, Z)) + \alpha(\eta(Z)g(\phi hX, Y) - \eta(Y)g(\phi hX, Z)).$$

By replacing X by hX in that equation, using (10.6) and the relation (6.1), we get

$$(10.18) \quad g((\nabla_X \phi)Y, Z) = (\eta(Z)g(hX, Y) - \eta(Y)g(hX, Z)) + \alpha(\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z)).$$

Then (10.12) follows from (10.18). On the other hand Eq. (10.12) can be written as

$$(\nabla_X \phi)Y = -g(\phi \mathcal{A}X, Y)\xi + \eta(Y)\phi \mathcal{A}X.$$

From (7.1) we find

$$(\nabla_X \phi h)Y - (\nabla_Y \phi h)X = R(X, Y)\xi - \alpha^2(\eta(X)Y - \eta(Y)X) - \alpha(\eta(X)\phi hY - \eta(Y)\phi hX).$$

Using (10.1) in the last equation we obtain (10.13). One can easily show that

$$(10.19a) \quad (\nabla_X \phi h)Y - (\nabla_Y \phi h)X = (\nabla_X \phi)hY - (\nabla_Y \phi)hX + \phi((\nabla_X h)Y - (\nabla_Y h)X).$$

By replacing Y by hY in (10.12), we get

$$(10.19b) \quad (\nabla_X \phi)hY = g(hY, hX + \alpha \phi X)\xi.$$

From (7.1) and (10.1), we have

$$(10.20) \quad \begin{aligned} & \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY) = \\ & \alpha^2(\eta(X)Y - \eta(Y)X) + \alpha(\eta(X)\phi hY - \eta(Y)\phi hX) + (\nabla_X \phi)Y - (\nabla_Y \phi)X. \end{aligned}$$

After using (10.19a) and (10.19b) in (10.20), we obtain

$$(10.21) \quad \begin{aligned} & \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY) = \\ & \alpha^2(\eta(X)Y - \eta(Y)X) + \alpha(\eta(X)\phi hY - \eta(Y)\phi hX) + \alpha g(\phi X, hY)\xi \\ & - \alpha g(\phi Y, hX)\xi + \phi((\nabla_X h)Y - (\nabla_Y h)X). \end{aligned}$$

Then by applying ϕ to (10.21) we have (10.14). \square

In the following result we additionally assume that the set $U : \alpha \neq 0$ is dense.

Theorem 12. *An almost α -para-Kenmotsu (κ, μ, ν) -spaces $(M^{2n+1}, \phi, \xi, \eta, g)$, satisfy the para-Kaehlerian structure condition.*

Proof. We only need to prove this if $n \geq 2$ for arbitrary 3-dimensional α -para-Kenmotsu manifold satisfies has para-Kaehler leaves. According to the Theorem 11 the α -paracosymplectic manifold $(U, \phi|_U, \xi|_U, \eta|_U, g|_U)$ is $D_{1,\alpha}$ -homotetic to almost para-Kenmotsu manifold $(U, \phi', \xi', \eta', g')$, (cf. 9.4). If necessary we restrict our attention to connected components of U . Now Eq.(10.12) tells us that U viewed as almost para-Kenmotsu manifold has para-Kaehler leaves. Moreover we notice that arbitrary $D_{\gamma,\beta}$ -homothety preserves this property thus we conclude that the original structure (ϕ, ξ, η, g) also satisfies the para-Kaehlerian structure condition. Finally if $(\nabla_X \phi)Y$ satisfies (10.12) on U , then this identity must be satisfied everywhere on M^{2n+1} . \square

For manifolds with constant sectional curvature c , $R(X, Y)\xi = c(\eta(Y)X - \eta(X)Y)$ thus in our terminology these manifolds are almost paracosymplectic $(c, 0, 0)$ -spaces.

Corollary 3. *An almost α -para-Kenmotsu manifold of constant sectional curvature has para-Kaehlerian leaves.*

Corollary 4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α -para-Kenmotsu (κ, μ, ν) -spaces. Then*

$$(10.22) \quad Q\phi - \phi Q = 2\mu h\phi X - 2(\nu + 2\alpha(1 - n))hX.$$

Proof. Using (10.1) and $\phi h = -h\phi$ we obtain $l\phi - \phi l = 2\mu h\phi X - 2\nu hX$. On the other hand from (10.11) one can easily prove that both $\eta \otimes \phi Q$ and $(\eta \circ Q\phi) \otimes \xi$ vanish. So (10.22) follows from (7.14). \square

Remark 3. *Manifolds which are conformal or locally conformal to cosymplectic manifolds were studied by many authors e.g.[10], [12], [21], [22], [29], [27].*

Remark 4. *$D_{\gamma,\beta}$ -homoteties as they appear in almost contact metric geometry are particular class of deformations considered by S. Tanno [38]. The general deformation of a metric (Riemannian) has a form $g' = \alpha g + \omega \otimes \theta + \theta \otimes \omega + \beta \omega \otimes \omega$ where ω, β are one-forms and α, β are functions, $\alpha > 0, \alpha + \beta > 0$. The paper seems to be nowadays completely forgotten in the framework of almost contact metric geometry.*

11. CLASSIFICATION OF THE 3-DIMENSIONAL ALMOST α -PARA-KENMOTSU
 (κ, μ, ν) -SPACES

In this section, different possibilities for the tensor field h are investigated. Thus we can comprehend the differences between the almost α -para-Kenmotsu and almost α -Kenmotsu cases by looking at the possible Jordan forms of the tensor field h .

It is well known that a self-adjoint linear operator Ψ of a Euclidean space is always diagonalizable, but this is not the case for a self-adjoint linear operator Ψ for a Lorentzian inner product. It is known ([35], pp. 50-55) that self-adjoint linear operator of a vector space with a Lorentzian inner product can be put into four possible canonical forms. In particular, the matrix representation g of the induced metric on M_1^3 is of Lorentz type, so the self-adjoint linear Ψ of M_1^3 can be put into one of the following four forms with respect to frames $\{e_1, e_2, e_3\}$ at $T_p M_1^3$ where $T_p M_1^3$ is a tangent space to M at p [26],[31].

$$\begin{aligned}
 \mathfrak{h}_1\text{-type)} \quad \Psi &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, & g &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 \mathfrak{h}_2\text{-type)} \quad \Psi &= \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, & g &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 \mathfrak{h}_3\text{-type)} \quad \Psi &= \begin{pmatrix} \gamma & -\lambda & 0 \\ \lambda & \gamma & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, & g &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda \neq 0, \\
 \mathfrak{h}_4\text{-type)} \quad \Psi &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix}, & g &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
 \end{aligned}$$

The matrices g for types $\mathfrak{h}_1)$ and $\mathfrak{h}_3)$ are with respect to an orthonormal basis of $T_p M_1^3$, whereas for types $\mathfrak{h}_2)$ and $\mathfrak{h}_4)$ are with respect to a pseudo-orthonormal basis. This is a basis $\{e_1, e_2, e_3\}$ of $T_p M_1^3$ satisfying $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$.

Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -paracosymplectic manifold. Then operator h has following types.

\mathfrak{h}_1 -type)

$$\begin{aligned}
 U_1 &= \{p \in M \mid h(p) \neq 0\} \subset M \\
 U_2 &= \{p \in M \mid h(p) = 0, \text{ in a neighborhood of } p\} \subset M
 \end{aligned}$$

That h is a smooth function on M implies $U_1 \cup U_2$ is an open and dense subset of M , so any property satisfied in $U_1 \cup U_2$ is also satisfied in M . For any point $p \in U_1 \cup U_2$ there exists a local orthonormal ϕ -basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p , where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. On U_1 we put $h e = \lambda e$, where λ is a non-vanishing smooth function. Since $tr h = 0$, we have $h \phi e = -\lambda \phi e$. The eigenvalue function λ is continuous on M and smooth on $U_1 \cup U_2$. So, h has following form

$$(11.1) \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respect to local orthonormal ϕ -basis $\{e, \phi e, \xi\}$.

\mathfrak{h}_2 -type) Using same methods in [25] one can construct a local pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ in a neighborhood of p where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$. Let \mathcal{U} be the open subset of M where $h \neq 0$. For every $p \in \mathcal{U}$ there exists an open neighborhood of p such that $he_1 = e_2, he_2 = 0, he_3 = 0$ and $\phi e_1 = \pm e_1, \phi e_2 = \mp e_2, \phi e_3 = 0$ and also $\xi = e_3$. Thus the tensor h has the form

$$(11.2) \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

relative a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$.

\mathfrak{h}_3 -type) We can find a local orthonormal ϕ -basis $\{e, \phi e, \xi\}$ in a neighborhood of p where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. Now, let \mathcal{U}_1 be the open subset of M where $h \neq 0$ and let \mathcal{U}_2 be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p . $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open subset of M . For every $p \in \mathcal{U}_1$ there exists an open neighborhood of p such that $he = \lambda \phi e, h\phi e = -\lambda e$ and $h\xi = 0$ where λ is a non-vanishing smooth function. Since $trh = 0$, the matrix form of h is given by

$$(11.3) \quad \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to local orthonormal basis $\{e, \phi e, \xi\}$.

\mathfrak{h}_4 -type) Then a local pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ is constructed in a neighborhood of p where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$. Since the tensor h is \mathfrak{h}_4 -type (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$) then $he_1 = \lambda e_1 + e_3, he_2 = \lambda e_2$ and $he_3 = e_2 + \lambda e_3$. Since $0 = trh = g(he_1, e_2) + g(he_2, e_1) + g(he_3, e_3) = 3\lambda$, then $\lambda = 0$. We write $\xi = g(\xi, e_2)e_1 + g(\xi, e_1)e_2 + g(\xi, e_3)e_3$ respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3\}$. Since $h\xi = 0$, we have $0 = g(\xi, e_2)e_3 + g(\xi, e_3)e_2$. Hence we get $\xi = g(\xi, e_1)e_2$ which leads to a contradiction with $g(\xi, \xi) = 1$. Thus, this case does not occur.

Since the proof of following lemma is similar to [25] we omit proof of it.

Lemma 5. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold. Then a canonical form of h stays constant in an open neighborhood of any point for h .*

In a 3-dimensional pseudo-Riemannian manifold case, the curvature tensor can be written by

$$(11.4) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y).$$

for any $X, Y, Z \in \Gamma(TM)$.

Using same procedure with [34], we have

Lemma 6. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold with h of \mathfrak{h}_1 type. Then for the covariant derivative on \mathcal{U}_1 the following equations are valid*

$$\begin{aligned}
i) \nabla_e e &= \frac{1}{2\lambda} [\sigma(e) - (\phi e)(\lambda)] \phi e + \alpha \xi, & ii) \nabla_e \phi e &= \frac{1}{2\lambda} [\sigma(e) - (\phi e)(\lambda)] e - \lambda \xi, \\
iii) \nabla_e \xi &= \alpha e + \lambda \phi e, \\
iv) \nabla_{\phi e} e &= -\frac{1}{2\lambda} [\sigma(\phi e) + e(\lambda)] \phi e - \lambda \xi, & v) \nabla_{\phi e} \phi e &= -\frac{1}{2\lambda} [\sigma(\phi e) + e(\lambda)] \phi e - \alpha \xi, \\
vi) \nabla_{\phi e} \xi &= \alpha \phi e - \lambda e \\
vii) \nabla_{\xi} e &= a_1 \phi e, & viii) \nabla_{\xi} \phi e &= a_1 e, \\
(11.5) [e, \xi] &= \alpha e + (\lambda - a_1) \phi e, & x) [\phi e, \xi] &= -(\lambda + a_1) e + \alpha \phi e, \\
xi) [e, \phi e] &= \frac{1}{2\lambda} [\sigma(e) - (\phi e)(\lambda)] e + \frac{1}{2\lambda} [\sigma(\phi e) + e(\lambda)] \phi e, \\
xii) \nabla_{\xi} h &= \xi(\lambda) s - 2a_1 h \phi, & xiii) h^2 - \alpha^2 \phi^2 &= \frac{1}{2} S(\xi, \xi) \phi^2.
\end{aligned}$$

where

$$a_1 = g(\nabla_{\xi} e, \phi e), \quad \sigma = S(\xi, \cdot)_{\ker \eta}.$$

Lemma 7. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold with h of \mathfrak{h}_1 type. Then the Ricci operator Q is given by*

$$(11.6) \quad Q = \left(\frac{r}{2} + \alpha^2 - \lambda^2\right) I + \left(-\frac{r}{2} + 3(\lambda^2 - \alpha^2)\right) \eta \otimes \xi - 2\alpha \phi h - \phi(\nabla_{\xi} h) + \sigma(\phi^2) \otimes \xi - \sigma(e) \eta \otimes e + \sigma(\phi e) \eta \otimes \phi e.$$

Lemma 8. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold with h of \mathfrak{h}_2 type. Then for the covariant derivative on \mathcal{U} the following equations are valid*

$$\begin{aligned}
i) \nabla_{e_1} e_1 &= -b_1 e_1 + \xi, & ii) \nabla_{e_1} e_2 &= b_1 e_2 - \alpha \xi, & iii) \nabla_{e_1} \xi &= \alpha e_1 - e_2, \\
iv) \nabla_{e_2} e_1 &= -b_2 e_1 - \alpha \xi, & v) \nabla_{e_2} e_2 &= b_2 e_2, & vi) \nabla_{e_2} \xi &= \alpha e_2, \\
vii) \nabla_{\xi} e_1 &= a_2 e_1, & viii) \nabla_{\xi} e_2 &= -a_2 e_2, \\
(11.7) ix) [e_1, \xi] &= (\alpha - a_2) e_1 - e_2, & x) [e_2, \xi] &= (\alpha + a_2) e_2, \\
xi) [e_1, e_2] &= b_2 e_1 + b_1 e_2. \\
xii) \nabla_{\xi} h &= -2a_2 h \phi, & xiii) h^2 &= 0.
\end{aligned}$$

where $a_2 = g(\nabla_{\xi} e_1, e_2)$, $b_1 = g(\nabla_{e_1} e_2, e_1)$ and $b_2 = g(\nabla_{e_2} e_2, e_1) = -\frac{1}{2} \sigma(e_1)$.

Proof. By $\nabla \xi = -\alpha^2 \phi + \phi h$, we obtain $iii), vi)$.

Using pseudo-orthonormal basis $\{e_1, e_2, e_3 = \xi\}$ with $\phi e_1 = e_1$, $\phi e_2 = -e_2$, $\phi e_3 = 0$ we have

$$\begin{aligned}
\nabla_{e_1} e_2 &= g(\nabla_{e_1} e_2, e_2) e_1 + g(\nabla_{e_1} e_2, e_1) e_2 + g(\nabla_{e_1} e_2, \xi) \xi \\
&= g(\nabla_{e_1} e_2, e_1) e_2 - g(e_2, \nabla_{e_1} \xi) \xi \\
&\stackrel{iii)}{=} g(\nabla_{e_1} e_2, e_1) e_2 - \alpha \xi \\
&= b_1 e_2 - \alpha \xi.
\end{aligned}$$

The proofs of other covariant derivative equalities are similar to $ii)$.

Putting $X = e_1$, $Y = e_2$ and $Z = \xi$ in the equation (11.4), we have

$$(11.8) \quad R(e_1, e_2) \xi = -\sigma(e_1) e_2 + \sigma(e_2) e_1.$$

On the other hand, by using (7.1), we get

$$(11.9) \quad \begin{aligned} R(e_1, e_2)\xi &= (\nabla_{e_1}\phi h)e_2 - (\nabla_{e_2}\phi h)e_1 \\ &= 2b_2e_2. \end{aligned}$$

Comparing (11.9) with (11.8), we obtain

$$(11.10) \quad \sigma(e_1) = -2b_2, \quad \sigma(e_2) = 0 = S(\xi, e_2).$$

Hence, the function b_2 is obtained from the last equation. \square

Lemma 9. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold with h of \mathfrak{h}_2 type. Then the Ricci operator Q is given by*

$$(11.11) \quad Q = \left(\frac{r}{2} + \alpha^2\right)I - \left(\frac{r}{2} + 3\alpha^2\right)\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h) + \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_2.$$

Proof. From (11.4), we obtain

$$R(X, \xi)\xi = S(\xi, \xi)X - S(X, \xi)\xi + QX - \eta(X)Q\xi - \frac{r}{2}(X - \eta(X)\xi),$$

for any vector field X . By (7.2) and (7.6) the last equation reduces to

$$(11.12) \quad QX = \frac{1}{2}S(\xi, \xi)\phi^2X - 2\alpha\phi hX - \phi(\nabla_\xi h)X - S(\xi, \xi)X + S(X, \xi)\xi + \eta(X)Q\xi + \frac{r}{2}(X - \eta(X)\xi).$$

By setting $S(X, \xi) = S(\phi^2X, \xi) + \eta(X)S(\xi, \xi)$ in (11.12), we have

$$(11.13) \quad QX = \frac{S(\xi, \xi)}{2}\phi^2X - 2\alpha\phi hX - \phi(\nabla_\xi h)X - S(\xi, \xi)X + S(\phi^2X, \xi)\xi + \eta(X)S(\xi, \xi)\xi + \eta(X)Q\xi + \frac{r}{2}\phi^2X.$$

On the other hand, the Ricci tensor S can be written with respect to the orthonormal basis $\{e_1, e_2, \xi\}$ as following

$$(11.14) \quad Q\xi = \sigma(e_1)e_2 + S(\xi, \xi)\xi$$

Using (11.14) in (11.13), we get

$$(11.15) \quad \begin{aligned} QX &= \frac{1}{2}(r + 2\alpha^2)X - \frac{1}{2}(6\alpha^2 + r)\eta(X)\xi - 2\alpha\phi h \\ &\quad - \phi(\nabla_\xi h)X + \sigma(\phi^2X)\xi + \eta(X)\sigma(e_1)e_2 + \end{aligned}$$

for arbitrary vector field X . This ends the proof. \square

Lemma 10. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold with h of \mathfrak{h}_3 type. Then for the covariant derivative on \mathcal{U}_1 the following equations are valid*

$$(11.16) \quad \begin{aligned} i) \nabla_e e &= b_3\phi e + (\alpha + \lambda)\xi, \quad ii) \nabla_e \phi e = b_3e, \quad iii) \nabla_e \xi = (\alpha + \lambda)e, \\ iv) \nabla_{\phi e} e &= b_4\phi e, \quad v) \nabla_{\phi e} \phi e = b_4e + (\lambda - \alpha)\xi, \quad vi) \nabla_{\phi e} \xi = -(\lambda - \alpha)\phi e, \\ vii) \nabla_\xi e &= a_3\phi e, \quad viii) \nabla_\xi \phi e = a_3e, \\ ix) [e, \xi] &= (\alpha + \lambda)e - a_3\phi e, \quad x) [\phi e, \xi] = -a_3e - (\lambda - \alpha)\phi e, \\ xi) [e, \phi e] &= b_3e - b_4\phi e, \\ xii) \nabla_\xi h &= \xi(\lambda)s - 2a_3h\phi, \quad xiii) h^2 - \alpha^2\phi^2 = \frac{1}{2}S(\xi, \xi)\phi^2. \end{aligned}$$

where $a_3 = g(\nabla_\xi e, \phi e)$, $b_3 = -\frac{1}{2\lambda}[\sigma(\phi e) + (\phi e)(\lambda)]$ and $b_4 = \frac{1}{2\lambda}[\sigma(e) - e(\lambda)]$.

Proof. By $\nabla\xi = \alpha\phi^2 + \phi h$, we have *iii*), *vi*).

Using ϕ -basis, we have

$$\begin{aligned}\nabla_\xi\phi e &= -g(\nabla_\xi\phi e, e)e + g(\nabla_\xi\phi e, \phi e)\phi e + g(\nabla_\xi\phi e, \xi)\xi \\ &= g(\phi e, \nabla_\xi e)e = a_3e,\end{aligned}$$

So we prove *viii*) . The proofs of other covariant derivative equalities are similar to *viii*).

Setting $X = e$, $Y = \phi e$, $Z = \xi$ in the equation (11.4), we have

$$R(e, \phi e)\xi = -g(Qe, \xi)\phi e + g(Q\phi e, \xi)e.$$

Since $\sigma(X) = g(Q\xi, X)$, we have

$$(11.17) \quad R(e, \phi e)\xi = -\sigma(e)\phi e + \sigma(\phi e)e.$$

On the other hand, by using (7.1), we have

$$\begin{aligned}(11.18) \quad R(e, \phi e)\xi &= (\nabla_e\phi h)\phi e - (\nabla_{\phi e}\phi h)e \\ &= (-2b_3\lambda - (\phi e)(\lambda))e + (-2b_4\lambda - e(\lambda))\phi e.\end{aligned}$$

Comparing (11.18) with (11.17), we get

$$\sigma(e) = e(\lambda) + 2b_4\lambda, \quad \sigma(\phi e) = -(\phi e)(\lambda) - 2b_3\lambda.$$

Hence, the functions b_3 and b_4 are obtained from the last equation. \square

Lemma 11. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold with h of \mathfrak{h}_3 type. Then the Ricci operator Q is given by*

$$(11.19) \quad Q = aI + b\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h) + \sigma(\phi^2) \otimes \xi - \sigma(e)\eta \otimes e + \sigma(\phi e)\eta \otimes \phi e,$$

where a and b are smooth functions defined by $a = \alpha^2 + \lambda^2 + \frac{r}{2}$ and $b = -3(\lambda^2 + \alpha^2) - \frac{r}{2}$, respectively.

Proof. Using (11.4), we get

$$R(X, \xi)\xi = S(\xi, \xi)X - S(X, \xi)\xi + QX - \eta(X)Q\xi - \frac{r}{2}(X - \eta(X)\xi),$$

for any vector field X . By (7.2), the last equation reduces to

$$(11.20) \quad QX = -\alpha^2\phi^2X + h^2X - 2\alpha\phi hX - \phi(\nabla_\xi h)X - S(\xi, \xi)X + S(X, \xi)\xi + \eta(X)Q\xi + \frac{r}{2}(X - \eta(X)\xi).$$

By writing $S(X, \xi) = S(\phi^2X, \xi) + \eta(X)S(\xi, \xi)$ in (11.20), we obtain

$$(11.21) \quad QX = \frac{S(\xi, \xi)}{2}\phi^2X - 2\alpha\phi hX - \phi(\nabla_\xi h)X - S(\xi, \xi)X + S(\phi^2X, \xi)\xi + \eta(X)S(\xi, \xi)\xi + \eta(X)Q\xi + \frac{r}{2}\phi^2X.$$

On the other hand S can be written with respect to the orthonormal basis $\{e, \phi e, \xi\}$ as

$$(11.22) \quad Q\xi = -\sigma(e)e + \sigma(\phi e)\phi e + S(\xi, \xi)\xi.$$

Using (11.22) in (11.21), we have

$$\begin{aligned}(11.23) \quad QX &= \left(\alpha^2 + \lambda^2 + \frac{r}{2}\right)X + \left(-3(\lambda^2 + \alpha^2) - \frac{r}{2}\right)\eta(X)\xi - 2\alpha\phi hX \\ &\quad - \phi(\nabla_\xi h)X + \sigma(\phi^2X)\xi - \eta(X)\sigma(e)e + \eta(X)\sigma(\phi e)\phi e,\end{aligned}$$

for arbitrary vector field X . This completes the proof. \square

Theorem 13. *Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α -para-Kenmotsu manifold. If the characteristic vector field ξ is harmonic map then almost α -paracosymplectic (κ, μ, ν) -manifold always exist on every open and dense subset of M . Conversely, if M is an almost α -paracosymplectic (κ, μ, ν) -manifold then the characteristic vector field ξ is harmonic map.*

Proof. We will prove theorem for three cases respect to chosen (pseudo) orthonormal basis.

Case 1: We assume that h is \mathfrak{h}_1 type.

Since ξ is a harmonic vector field, ξ is an eigenvector of Q . Hence we deduce that $\sigma = 0$. Putting $s = \frac{1}{\lambda}h$ in (11.5) xii), we find

$$(11.24) \quad Q = \left(\frac{r}{2} + \alpha^2 - \lambda^2\right)I + \left(-\frac{r}{2} + 3(\lambda^2 - \alpha^2)\right)\eta \otimes \xi - 2a_1h - \left(2\alpha + \frac{\xi(\lambda)}{\lambda}\right)\phi h.$$

Setting $Z = \xi$ in (11.4) and using (11.24), we obtain

$$R(X, Y)\xi = (-\alpha^2 + \lambda^2)(\eta(Y)X - \eta(X)Y) - 2a_1(\eta(Y)hX - \eta(X)hY) - \left(2\alpha + \frac{\xi(\lambda)}{\lambda}\right)(\eta(Y)\phi hX - \eta(X)\phi hY),$$

where the functions κ , μ and ν defined by $\kappa = \frac{S(\xi, \xi)}{2} = (\lambda^2 - \alpha^2)$, $\mu = -2a_1$, $\nu = -(2\alpha + \frac{\xi(\lambda)}{\lambda})$, respectively. Moreover, using (11.24), we have $Q\phi - \phi Q = 2\mu h\phi - 2\nu h$.

Case 2: Secondly, let h be \mathfrak{h}_2 type.

Putting $\sigma = 0$ in (11.11) and using (11.7) xii) we get

$$(11.25) \quad Q = \left(\frac{r}{2} + \alpha^2\right)I - \left(\frac{r}{2} + 3\alpha^2\right)\eta \otimes \xi - 2a_2h - 2\alpha\phi hX.$$

When $\xi = Z$ in (11.4) we obtain

$$(11.26) \quad \begin{aligned} R(X, Y)\xi &= -S(X, \xi) + S(Y, \xi) - \eta(X)QY \\ &\quad + \eta(Y)QX + \frac{r}{2}(\eta(X)Y - \eta(Y)X), \end{aligned}$$

for any vector fields X, Y . By applying (11.25) in (11.26), we have

$$R(X, Y)\xi = -\alpha^2(\eta(Y)X - \eta(X)Y) - 2a_2(\eta(Y)hX - \eta(X)hY) - 2\alpha(\eta(Y)\phi hX - \eta(X)\phi hY)$$

where the functions κ , μ and ν defined by $\kappa = \frac{S(\xi, \xi)}{2} = -\alpha^2$, $\mu = -2a_2$, $\nu = -2\alpha$, respectively. Furthermore, by (11.25), we have $Q\phi - \phi Q = 2\mu h\phi - 2\nu h$.

Case 3: Finally, we suppose that h is \mathfrak{h}_3 type.

Since ξ is a harmonic map, we have $\sigma = 0$. Putting $s = \frac{1}{\lambda}h$ in (11.19) we get

$$(11.27) \quad Q = aI + b\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h),$$

Setting $\xi = Z$ in (11.4) we again obtain

$$(11.28) \quad \begin{aligned} R(X, Y)\xi &= -S(X, \xi) + S(Y, \xi) - \eta(X)QY \\ &\quad + \eta(Y)QX + \frac{r}{2}(\eta(X)Y - \eta(Y)X), \end{aligned}$$

for any vector fields X, Y . Using (11.27) in (11.28), we get

$$R(X, Y)\xi = -(\alpha^2 + \lambda^2)(\eta(Y)X - \eta(X)Y) - 2a_3(\eta(Y)hX - \eta(X)hY) - \left(2\alpha + \frac{\xi(\lambda)}{\lambda}\right)(\eta(Y)\phi hX - \eta(X)\phi hY),$$

where the functions κ , μ and ν are defined by $\kappa = -(\alpha^2 + \lambda^2)$, $\mu = -2a_3$, $\nu = -(2\alpha + \frac{\xi(\lambda)}{\lambda})$, respectively. By help of (11.27), we get $Q\phi - \phi Q = 2\mu h\phi - 2\nu h$.

This completes the proof. \square

12. EXAMPLE

Example 1. We consider the 3-dimensional manifold

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0, y \neq 0\}$$

and the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \phi e_1 = \frac{\partial}{\partial y}, \quad e_3 = \xi = x \frac{\partial}{\partial x} + (y + 2x) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dz$ and the fundamental 2-form $\Phi = dx \wedge dy - (y + 2x)dx \wedge dz - xdy \wedge dz$ defines an almost para-Kenmotsu manifold.

Let g, ϕ be the pseudo-Riemannian metric and the $(1, 1)$ -tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -x \\ 0 & -1 & y + 2x \\ -x & y + 2x & 1 - 3x^2 - 4xy - y^2 \end{pmatrix},$$

$$\phi = \begin{pmatrix} 0 & 1 & -(y + 2x) \\ 1 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix}.$$

We easily get

$$\begin{aligned} [e_1, e_2] &= 0, \\ [e_1, e_3] &= e_1 + 2e_2, \\ [e_2, e_3] &= e_2. \end{aligned}$$

Moreover, the above example is an almost para-Kenmotsu $(\kappa, \mu, \nu) = (1, 1, -2)$ -space.

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